A geometric model of twisted differential K-theory

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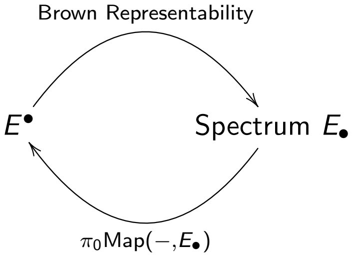
Outline

Definition

A **generalized cohomology theory** is a functor $E^\bullet : \mathbf{Top}_*^{\mathrm{op}} \to \mathbf{GrAb}$ satisfying:

- Wedge axiom
- Mayer-Vietoris property
- Homotopy invariance



Example (Examples of Geometric cocycles)

- $H^\bullet_{\text{sing}}(-;\mathbb{Z})$: integral cochains
- K^0: complex vector bundles

Let $E^\bullet$ be a generalized cohomology theory (such as elliptic cohomologies, TMF, Morava K-theory, $\cdots$) and X a space.

- **Question:** Can we represent an element of $E^n(X)$ using geometric objects (in X, over X, ...)?

Differential cohomology theories

The idea

On a smooth manifold, there are

- Topological data — spectrum E
- Differential form data — de Rham complex $\Omega^\bullet \otimes_{\mathbb{R}} A$

The idea of differential cohomology theory is to combine them in a homotopy theoretic way.

Differential cohomology theories

The Hopkin-Singer Model

Hopkins and Singer (2002): Given any cohomology theory $E^\bullet$ and a fixed sequence of cocycles $c = (c_n)$ representing universal characteristic classes, there exists a differential extension $\widehat{E}^\bullet$.

Geometric models of differential K-theory

- Freed-Lott-Klonoff triple model (Klonoff 2008 and Freed-Lott 2009)
  - Cycle: (E, ∇, ω)
  - Equivalence relation: $(E, \nabla_E, \omega_E) \sim (F, \nabla_F, \omega_F)$ iff there exists (G, ∇_G) and an isomorphism $\varphi : E \oplus G \to F \oplus G$ such that

$$\mathrm{cs}(t \mapsto (1-t)\nabla_E \oplus \nabla_G + t\varphi^*(\nabla_F \oplus \nabla_G)) \mod \mathrm{Im}(d) = \omega_E - \omega_F$$

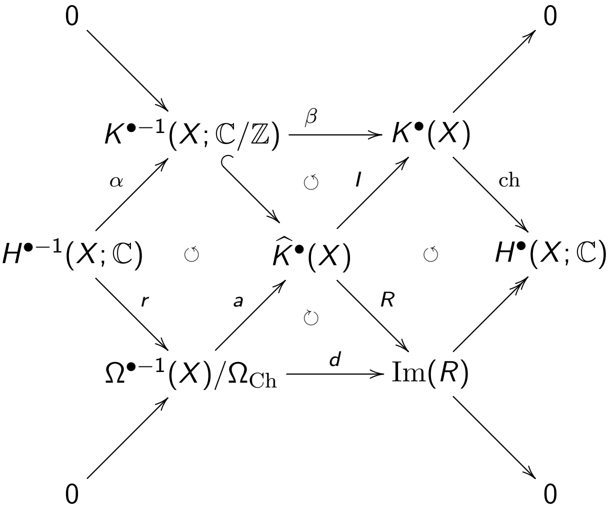
  - Monoid structure: $(\oplus, \oplus, +)$.

We obtain a commutative monoid $\mathfrak{M}$ of isomorphism classes.

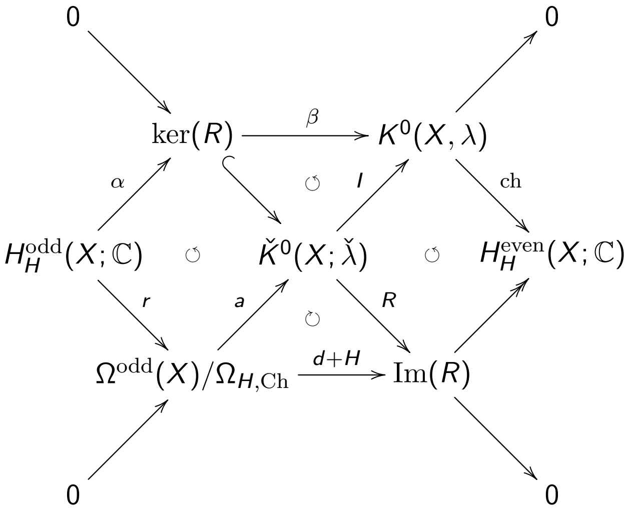
$$\widehat{K}^0_{\mathsf{FLK}}(X) := K(\mathfrak{M})$$

Differential K-theory

Hexagon diagram



Twisted differential K-theory hexagon diagram (P. 2016)



Definition (Karoubi, Bouwknegt et al (BCMMS), Waldorf, ...)

- $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X
- λ: a $U(1)$-valued completely normalized Čech 2-cocycle.

A λ-**twisted vector bundle** E over X:

- A family of product bundles $\{U_i \times \mathbb{C}^n : U_i \in \mathcal{U}\}_{i\in\Lambda}$
- Transition maps

  $$g_{ji} : U_{ij} \to U(n)$$

  satisfying

  $$g_{ii} = \mathbf{1}, \quad g_{ji} = g_{ij}^{-1}, \quad g_{kj}g_{ji} = g_{ki}\lambda_{kji}.$$

Definition (Karoubi, Bouwknegt et al (BCMMS), ...)

The **twisted K-theory** of X defined on an open cover $\mathcal{U}$ with a $U(1)$-gerbe twisting λ.

$$K^0(\mathcal{U}, \lambda) := K(\mathtt{Iso}(\mathbf{Bun}(\mathcal{U}, \lambda), \oplus)).$$

Definition (Karoubi, Bouwknegt et al (BCMMS), ...)

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Remark (No twisted vector bundle admits a nontorsion twist)

If λ represents a nontrivial non-torsion class in $H^2(\mathcal{U}, U(1))$, then there does not exist a finite rank λ-twisted vector bundle. (Consider $g_{ik}g_{kj}g_{ji} = \lambda_{kji}\mathbf{1}_n$ and take det.)

Definition

X: a manifold, $\mathcal{U} := \{U_i\}_{i\in\Lambda}$ an open cover of X.

- A $U(1)$-**gerbe** over X on $\mathcal{U}$: $\{\lambda_{kji}\} \in \check{Z}^2(\mathcal{U}, U(1))$
- A **connection** on a $U(1)$-gerbe $\{\lambda_{kji}\}$ on $\mathcal{U}$ is a pair $(\{A_{ji}\}, \{B_i\})$
  - $\{A_{ji} \in \Omega^1(U_{ij}; i\mathbb{R})\}_{i,j\in\Lambda}$
  - $\{B_i \in \Omega^2(U_i; i\mathbb{R})\}_{i\in\Lambda}$,

  such that the triple $\widehat{\lambda} := (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ is a 2-cocycle in Čech-de Rham double complex.

One of the cocycle conditions for $\widehat{\lambda}$: $B_j - B_i = dA_{ji}$

Definition

The 3-curvature H of $\widehat{\lambda}$ is defined by $H|_{U_i} := dB_i$.

Definition

- $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$
- $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle

A **connection** on E compatible with $\widehat{\lambda}$ is a family $\Gamma = \{\Gamma_i \in (\Omega^1(U_i; \mathfrak{u}(n)))\}_i$ satisfying that

$$\Gamma_i - g_{ji}^{-1}\Gamma_j g_{ji} - g_{ji}^{-1} dg_{ji} = -A_{ji} \cdot \mathbf{1}.$$

Definition

- $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$
- $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle
- Γ a connection on E compatible with $\widehat{\lambda}$

The **curvature form** of Γ is the family $R = \{R_i \in M_n(\Omega^2(U_i; \mathbb{C}))\}_i$, where $R_i := d\Gamma_i + \Gamma_i \wedge \Gamma_i$.

Proposition

For each $m \in \mathbb{Z}^+$, the differential forms $\mathrm{tr}\,[(R_i - B_i \cdot \mathbf{1})^m]$ over the open sets U_i glue together to define a global differential form on X.

Definition

- $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$
- $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle
- Γ a connection on E compatible with $\widehat{\lambda}$
- H is the 3-curvature of $\widehat{\lambda}$

The m^{th} **twisted Chern character form** is defined by

$$\mathrm{ch}_{(m)}(\Gamma) := \mathrm{tr}(R_i - B_i \cdot \mathbf{1})^m.$$

The **total twisted Chern character form** is defined by

$$\mathrm{ch}(\Gamma) := \mathrm{rank}(E) + \sum_{m=1}^{\infty} \frac{1}{m!} \mathrm{ch}_{(m)}(\Gamma).$$

Interlude: Twisted de Rham cohomology

X a smooth manifold, H is a closed 3-form.

- The **twisted de Rham complex.** The $\mathbb{Z}_2$-graded sequence of differential forms

$$\cdots \to \Omega^{\text{even}}(X) \xrightarrow{d+H} \Omega^{\text{odd}}(X) \xrightarrow{d+H} \cdots$$

  is a complex.
- The **twisted de Rham cohomology** of X is the cohomology of this complex, and denote it by $H_H^\bullet(X)$.
- If closed 3-forms H and H' are cohomologous, i.e. $H' = H + d\xi$, the multiplication by $\exp(\xi)$ induces an isomorphism $H_H^\bullet(X) \to H_{H'}^\bullet(X)$.

Chern-Weil theory of twisted vector bundles

Twisted Chern character forms — Properties

The total twisted Chern character form $\mathrm{ch}(\Gamma)$ is

- $(d + H)$-closed
- Additive under $\oplus$
- Natural
- Invariance/covariance under change of twists

Change of twists

- $\widehat{\lambda}_1 \xrightarrow{\widehat{\alpha}} \widehat{\lambda}_2$ with $\widehat{\lambda}_2 = \widehat{\lambda}_1 + D\widehat{\alpha}$, where $\widehat{\alpha} = (\{\chi_{ji}\}, \{\Pi_i\}) \in \check{C}^1(\mathcal{U}, \Omega^1)$
- $\widehat{\lambda}_1 = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\}) \xrightarrow{\xi} \widehat{\lambda}_2 = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i + \xi_i\})$, where $\xi \in \Omega^2(X; i\mathbb{R})$ and $\xi_i := \xi|_{U_i}$.

Definition

- $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$
- $E = (\mathcal{U}, \{g_{ji}\}, \{\lambda_{kji}\})$ be a λ-twisted vector bundle
- $\gamma : t \mapsto \Gamma_t$ be a path of connections on E such that each Γ_t is compatible with $\widehat{\lambda}$.
- $p : X \times I \to X$ is the projection map
- $\widetilde{\Gamma}$ is the connection on p^*E defined by $\widetilde{\Gamma}(x, t) = (p^*\Gamma_t)(x, t)$

The **twisted Chern-Simons form** of γ is the integration along the fiber:

$$\mathrm{cs}(\gamma) := \int_I \mathrm{ch}(\widetilde{\Gamma}) \in \Omega^{\mathrm{odd}}(X; \mathbb{C}).$$

Proposition

- $\mathrm{cs}(\gamma)$ is a transgression form.

$$(d + H)\mathrm{cs}(\gamma) = \mathrm{ch}(\Gamma_1) - \mathrm{ch}(\Gamma_0).$$

- $\mathrm{cs}(\gamma)$ of a loop is in the image of $d + H$.

Definition

The **twisted total Chern character** of E, denoted by $\mathrm{ch}(E)$, is the twisted cohomology class of $\mathrm{ch}(\Gamma)$ for any connection Γ on E.

Proposition

The assignment

$$\begin{aligned} \mathrm{ch} : K^0(\mathcal{U}, \lambda) &\to H_H^{\text{even}}(X; \mathbb{C}) \\ [E] - [F] &\mapsto [\mathrm{ch}(\Gamma^E)] - [\mathrm{ch}(\Gamma^F)], \end{aligned}$$

with $(\{A_{ji}\}, \{B_i\})$ a representative connection on λ and Γ^E and Γ^F representative connections on λ-twisted vector bundles E and F, respectively, both compatible with $\widehat{\lambda}$, is a well-defined group homomorphism called the **twisted Chern character**.

Twisted differential K-theory

History

- '07 Carey, Mickelsson, and Wang
  - Twisted differential K^{-1}-theory.
  - Choices: open cover, spectral cut, partition of unity
- '09 Kahle and Valentino: Proposed a list of axioms of twisted differential K-theory
- '14 Bunke and Nikolaus
  - Homotopy pullback in $\mathbf{Sp}_\infty(\mathbf{Mfld}/M)$
- '16 (Feb) P.
- '16 (Apr) Lott and Gorokhovsky
- '16 (May) Grady and Sati — AHSS in differential cohomology
- '17+ Grady and Sati — AHSS in twisted differential cohomology

Twisted differential K-theory — Axioms of Kahle and Valentino

Axioms on differential twists

X: a smooth manifold.

Axiom (Kahle and Valentino 2009 Section A.3.)

A **twisted differential even K-group** with a differential twist $\widehat{\lambda} \in \mathfrak{Twist}_{\widehat{K}}(X)$ is a group $\widehat{K}^0(X, \widehat{\lambda})$ satisfying the following axioms:

- **Existence of differential twist.** For each $X \in \mathbf{Man}$ there is a groupoid $\mathfrak{Twist}_{\widehat{K}}(X)$ consisting of geometric central extensions.
- **Forgetful and curvature functors.** There exist natural functors:

$$\begin{aligned} F :& \mathfrak{Twist}_{\widehat{K}}(X) \to \mathfrak{Twist}_K(X) \\ \mathrm{Curv} :& \mathfrak{Twist}_{\widehat{K}}(X) \to \Omega^3_{\mathrm{cl}}(X; \mathbb{R}). \end{aligned}$$

- **A twisted Chern character map.** For each $\widehat{\lambda} \in \mathfrak{Twist}_{\widehat{K}}(X)$, there exists a twisted Chern character map

$$\mathrm{ch}_{\widehat{\lambda}} : K^0(X, F(\widehat{\lambda})) \to H^{\mathrm{even}}_{\mathrm{Curv}(\widehat{\lambda})}(X; \mathbb{R})$$

natural with respect to pullback.

Given any differential twist $\widehat{\lambda} \in \mathfrak{Twist}_K(X)$, one may associate an abelian group $\widehat{K}^0(X, \widehat{\lambda})$ satisfying the following properties:

- **Functoriality.** For any smooth map $f : X \to Y$, we have an induced group homomorphism

$$f^* : \widehat{K}^0(Y, \widehat{\lambda}) \to \widehat{K}^0(X, f^*\widehat{\lambda}).$$

- **Naturality of twists.** For any morphism $\alpha \in \mathrm{Hom}_{\mathfrak{Twist}_{\widehat{K}}(X)}(\widehat{\lambda}, \widehat{\lambda}')$, there is a natural isomorphism

$$\phi_\alpha : \widehat{K}^0(X, \widehat{\lambda}) \xrightarrow{\cong} \widehat{K}^0(X, \widehat{\lambda}')$$

which is compatible with the "a" map and the R map.

- **Pull-back square.** There are natural transformations (for pullback along smooth maps and along isomorphism of twists)

$$\begin{aligned} I :& \widehat{K}^0(X,\widehat{\lambda}) \to \widehat{K}^0(X, F(\widehat{\lambda})) \\ R :& \widehat{K}^0(X,\widehat{\lambda}) \to \Omega^{\text{even}}_{\text{Curv}(\widehat{\lambda})}(X;\mathbb{R}) \\ a :& \Omega^{\text{odd}}_{\text{Curv}(\widehat{\lambda})}(X;\mathbb{R})/\text{Im}(\text{ch}) \to \widehat{K}^0(X,\widehat{\lambda}) \end{aligned}$$

satisfying that

$$R \circ a = d + \text{Curv}(\widehat{\lambda}) \qquad \text{and} \qquad \text{ch}_{\widehat{(\lambda)}} \circ I = \text{pr} \circ R$$

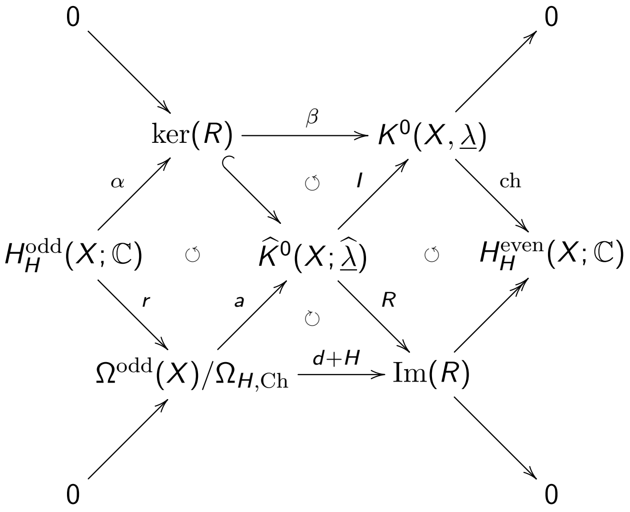
where $\text{pr} : \Omega^{\text{even}}_{\text{Curv}(\widehat{\lambda})}(X;\mathbb{R}) \to H^{\text{even}}_{\text{Curv}(\widehat{\lambda})}(X;\mathbb{R})$ is the canonical map taking de Rham cohomology class.

- **Exact sequences.** The following natural exact sequence holds:

$$0 \to \Omega^{\mathrm{odd}}_{\mathrm{Curv}(\widehat{\lambda})}(X;\mathbb{R})/\mathrm{Im}(\mathrm{ch}) \xrightarrow{a} \widehat{K}^0(X,\widehat{\lambda}) \xrightarrow{I} \widehat{K}^0(X,F(\widehat{\lambda})) \to 0$$

$$0 \to K^0(X,F(\widehat{\lambda});\mathbb{R}/\mathbb{Z}) \to \widehat{K}^0(X,\widehat{\lambda}) \xrightarrow{R} \Omega^{\mathrm{even}}_{\mathrm{Curv}(\widehat{\lambda})}(X;\mathbb{R})$$

Twisted differential K-theory hexagon diagram (P. 2016)



The **torsion differential** K**-twists** for an open cover $\mathcal{U}$ of X, denoted by $\texttt{Twist}^{\mathrm{tor}}_{\widehat{K}}(\mathcal{U})$, is a groupoid such that

- **objects** $\widehat{\lambda} = (\{\lambda_{kji}\}, \{A_{ji}\}, \{B_i\})$ with $[\lambda] \in \mathrm{Tor}(H^3(X;\mathbb{Z}))$.
- $\mathrm{Hom}(\widehat{\lambda}_1, \widehat{\lambda}_2) = \{(\widehat{\alpha}, \xi) \in \check{C}^1(\mathcal{U};\Omega^0) \oplus \check{C}^0(\mathcal{U};\Omega^1) \oplus \Omega^2(X; i\mathbb{R})) : \widehat{\lambda}_2 = \widehat{\lambda}_1 + D\widehat{\alpha} + \xi\}$

- **Cycles:** A $\widehat{K}^0(\mathcal{U};\widehat{\lambda})$-**generator** is a triple (E,Γ,ω) consisting of a λ-twisted vector bundle E defined on an open cover $\mathcal{U}=\{U_i\}_{i\in\Lambda}$ on X, a connection Γ on E compatible with $\widehat{\lambda}$, and $\omega\in\Omega^{\mathrm{odd}}(X;\mathbb{C})/\mathrm{Im}(d+H)$.
- **Equivalence relation:** Two $\widehat{K}^0(\mathcal{U};\widehat{\lambda})$-generators (E,Γ,ω) and (E',Γ',ω') are **equivalent** if there exists a λ-twisted vector bundle with connection (F,Γ^F) and a λ-twisted vector bundle isomorphism $\varphi=\{\varphi_i\}_{i\in\Lambda}:E\oplus F\to E'\oplus F$ such that $\mathrm{CS}(\Gamma\oplus\Gamma^F,\varphi^*(\Gamma'\oplus\Gamma^F))=\omega-\omega'$.
- **Monoid structure:** $(\oplus,\oplus,+)$

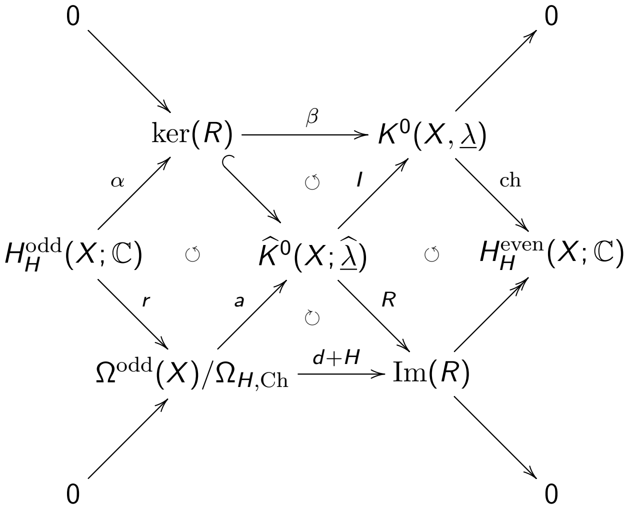
The set of isomorphism classes of $\widehat{K}^0(\mathcal{U};\widehat{\lambda})$-generators form a commutative monoid.

Definition (P. 2016)

- Let $\widehat{\lambda} \in \mathtt{Twist}^{\mathrm{tor}}_{\widehat{K}}(\mathcal{U})$. The **twisted differential K-group** $\widehat{K}^0(\mathcal{U}, \widehat{\lambda})$ is the Grothendieck group of the commutative monoid of isomorphism classes of $\widehat{K}^0(\mathcal{U}; \widehat{\lambda})$-generators.
- The **twisted differential K-group** of X, denoted by $\widehat{K}^0(X, \underline{\widehat{\lambda}})$, is defined by the colimit of $\widehat{K}^0(\mathcal{U}, \widehat{\lambda})$ over all refinements of $\mathcal{U}$.

A geometric model of twisted differential K-theory

Twisted differential K-theory hexagon diagram



Twisted differential K-theory

Status and Progress

- '17+ P. Non-torsion case using GL_1-bundle gerbe modules with connection.
- '17+ P. and Corbett Redden: The odd case.
- '17+ P. and Corbett Redden: Classification of equivariant gerbe connections

Thank you!

A detailed preprint is available on ArXiv.
`arXiv:1602.02292 [math.KT]`.

Appendix

The twisted odd Chern character

The category $\mathcal{P}(\mathcal{U}, \lambda)$

- Objects: (E, ϕ) where $E \in \mathbf{Bun}(\mathcal{U}, \lambda)$ and $\phi \in \mathrm{Aut}(E)$
- A morphism $\psi : (E, \phi) \to (E', \phi')$: A λ-twisted vector bundle isomorphism $\psi : E \to E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E' \\ \downarrow{\scriptstyle \phi} & \circlearrowright & \downarrow{\scriptstyle \phi'} \\ E & \xrightarrow{\psi} & E' \end{array}$$

Definition

The **twisted K_1-group** $K_1(\mathcal{U}, \lambda)$ is the free abelian group generated by $\mathtt{Isom}(\mathcal{P}(\mathcal{U}, \lambda))$ modulo the following relations:

(1) $(E_1 \oplus E_2, \phi_1 \oplus \phi_2) = (E_1, \phi_1) + (E_2, \phi_2)$.

(2) $(E, \phi_1 \circ \phi_2) = (E, \phi_1) + (E, \phi_2)$.

Let $\lambda = \{\lambda_{kji}\}$ a torsion $U(1)$-gerbe on $\mathcal{U}$.

Definition

The **twisted odd Chern character** is the map

$$\begin{aligned} \mathrm{Ch} : K_1(X,\lambda) &\to H_H^{\mathrm{odd}}(X;\mathbb{C}) \\ (E,\phi) &\mapsto \left[\mathrm{cs}\left(t \mapsto (1-t)\Gamma^E + t\phi^*\Gamma^E\right)\right], \end{aligned}$$

where Γ^E is a connection on E compatible with $(\{\lambda_{kji}\},\{A_{ji}\},\{B_i\})$ for some connection $(\{A_{ji}\},\{B_i\})$ on the $U(1)$-gerbe λ that has the 3-curvature H.

Appendix

The twisted odd Chern character

$\mathrm{Ch}(E, \phi)$

- Represents an odd twisted cohomology class.
- Well-defined on the isomorphism classes.
- Independent of choices of connection Γ.
- Invariant/covariant under the change of connections on $\widehat{\lambda}$.
- Additive under $(\oplus, \oplus)$
- Additive under $(\mathbf{1}, \circ)$
- Functorial