A classification of equivariant gerbe connections

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Outline

Differential Cohomology

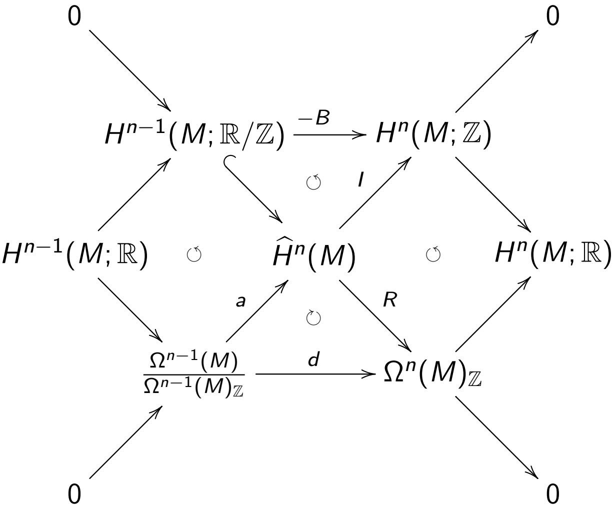
Hopkins-Singer Cochain Model

$$\begin{array}{ccc} \widehat{C}^n(M) & \longrightarrow & \Omega^n(M;\mathbb{R}) \\ \downarrow & & \downarrow \\ C^n(M;\mathbb{Z}) & \longrightarrow & C^n(M;\mathbb{R}) \end{array}$$

- $\widehat{C}^n(M) = C^n(M;\mathbb{Z}) \times C^{n-1}(M;\mathbb{R}) \times \Omega^n(M;\mathbb{R})$
- $\widehat{d}(c,h,\omega) = (\delta c, c - \int \omega - \delta h, d\omega)$
- $\widehat{H}^\bullet(M) := \mathrm{Ker}(\widehat{d})/\mathrm{Im}(\widehat{d})$.

Differential Cohomology

Hexagon diagram à la Cheeger and Simons



What is a gerbe?

Geometric way to represent elements of $H^2(M;\mathbb{Z})$

M: a smooth manifold.

There are geometric cocycles representing elements of $H^2(M;\mathbb{Z})$.

(L,∇) a complex line bundle over M with connection ∇.

$c_1(\nabla) := \frac{i}{2\pi}F_\nabla$: the first *Chern class* of ∇.

Chern-Weil theorem says $c_1(\nabla) \in \Omega^2(M;\mathbb{C})$ is a topological invariant of a line bundle L. i.e.

$c_1(\nabla) - c_1(\nabla') = d\mathrm{cs}(t \mapsto ((1-t)\nabla + t\nabla'))$.

$c_1(L) \in H^2_{\mathrm{dR}}(M;\mathbb{C})$ and in fact $c_1(L) \in H^2(M;\mathbb{Z})$

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Question

How do we geometrically represent elements of $H^n(M;\mathbb{Z})$ for $n = 3, 4, \ldots$? What is the higher analogue of line bundles?

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Answer: $(n-2)$-gerbes

What is a gerbe?

Hierarchy of $(n-2)$-gerbes with connection and differential cohomology

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \\
C^2(\mathcal{U}, \underline{S^1}) & \xrightarrow{d} & C^2(\mathcal{U}, \Omega^1_M) & \xrightarrow{d} & C^2(\mathcal{U}, \Omega^2_M) & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \\
C^1(\mathcal{U}, \underline{S^1}) & \xrightarrow{d} & C^1(\mathcal{U}, \Omega^1_M) & \xrightarrow{d} & C^1(\mathcal{U}, \Omega^2_M) & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \\
C^0(\mathcal{U}, \underline{S^1}) & \xrightarrow{d} & C^0(\mathcal{U}, \Omega^1_M) & \xrightarrow{d} & C^0(\mathcal{U}, \Omega^2_M) & \xrightarrow{d} & \cdots
\end{array}
$$

$$\widehat{H}^1(M) \cong C^\infty(M, S^1)$$

$$\widehat{H}^2(M) \cong \mathrm{Bun}_{S^1,\nabla}(M)_{/\cong}$$

$$\widehat{H}^3(M) \cong \mathrm{Grb}_\nabla(M)_{/\cong}$$

What is a gerbe?

Example: An obstruction for lifting a principal bundle

$$1 \to A \to \widetilde{G} \to G \to 1 \text{ a central extension of } G.$$

Question

Let $P \to M$ be a principal G-bundle. What is the necessary and sufficient condition for lifting a principal G-bundle P to a principal $\widetilde{G}$-bundle $\widetilde{P}$?

What is a gerbe?

Example: An obstruction for lifting a principal bundle

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Question

Let $P \to M$ be a principal G-bundle. What is the necessary and sufficient condition for lifting a principal G-bundle P to a principal $\widetilde{G}$-bundle $\widetilde{P}$?

Answer: Choose a good cover $\mathcal{U} = \{U_i\}$ of M. The transition maps $g_{ij} : U_{ij} \to G$ are lifted to $\widetilde{g}_{ij} : U_{ij} \to \widetilde{G}$. Define $\lambda_{ijk} := \widetilde{g}_{ij}\widetilde{g}_{jk}\widetilde{g}_{ki}$. So P lifts to $\widetilde{P}$ if and only if

- λ_{ijk} represents $1 \in \check{H}^2(\mathcal{U}, A)$.
- The gerbe with cocycle λ is trivializable.

S^1-bundle gerbes with connection

Definition, characteristic class

Definition (Murray '94)

Let $M \in \mathbf{Man}$. An object $\mathcal{L} = (L, \mu, \pi) \in \mathrm{Grb}(M)$ consists of the following:

- A surjective submersion $Y \xrightarrow{\pi} M$.
- $L \in \mathrm{Bun}_{S^1}(Y^{[2]})$. (Here $Y^{[2]} = Y \times_M Y$.)
- Isomorphism $\mu\colon \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L$ in $\mathrm{Bun}_{S^1}(Y^{[3]})$ that is associative over $Y^{[4]}$.

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After choosing an open cover $\mathcal{U} = \{U_i\}$, the isomorphism μ determines a degree 2 Čech cochain $g_{ijk} : U_{ijk} \to S^1$. The associativity of μ is the condition for g being a cocycle. Hence we obtain a class in $H^3(M; \mathbb{Z})$ which is called the **Diximier-Douady class** of $\mathcal{L}$

S^1-bundle gerbes with connection

Definition

Definition (Murray '94)

Let $M \in \mathbf{Man}$. An object $\widehat{\mathcal{L}} = (L, \mu, \pi, \nabla, B) \in \mathrm{Grb}_\nabla(M)$ consists of the following:

- A surjective submersion $Y \xrightarrow{\pi} M$.
- $(L, \nabla) \in \mathrm{Bun}_{S^1,\nabla}(Y^{[2]})$. (Here $Y^{[2]} = Y \times_M Y$.)
- Isomorphism $\mu \colon \pi_{12}^*(L, \nabla) \otimes \pi_{23}^*(L, \nabla) \to \pi_{13}^*(L, \nabla)$ in $\mathrm{Bun}_{S^1,\nabla}(Y^{[3]})$ that is associative over $Y^{[4]}$.
- $B \in \Omega^2(Y)$ satisfying $F_\nabla = \pi_2^* B - \pi_1^* B =: \delta B \in \Omega^2(Y^{[2]})$.

S^1-bundle gerbes with connection

The 3-curvature

- Since $0 = dF_\nabla = d\delta B = \delta dB$ and the sequence

$$0 \to \Omega^3(M) \xrightarrow{\pi^*} \Omega^3(Y) \xrightarrow{\delta} \Omega^3(Y^{[2]}) \xrightarrow{\delta} \cdots$$

  is an acyclic complex, there exists a unique $H \in \Omega^3_{\text{closed}}(M)$ such that $\pi^* H = dB$. This closed 3-form is called the 3-**curvature** of bundle gerbe.

- The de Rham cohomology class $[H/2\pi i] \in H^3_{\text{dR}}(M; \mathbb{R})$ is in fact an integral cohomology class equal to the Diximier-Douady class of $\mathcal{L} = (L, \mu, \pi)$.

Interlude: Equivariant cohomology and Cartan model of G-equivariant de Rham complex

From now on, G denotes a compact Lie group acting on a smooth manifold M.

Definition (Borel equivariant cohomology)

Let A be any abelian group. $H^\bullet_G(M; A) := H^\bullet(EG \times_G M; A)$.

Definition (Cartan model)

The G-equivariant de Rham complex is a complex $(\Omega^\bullet_G, d_G)$ where

$$\Omega^k_G(M) := \bigoplus_{k=2i+j} \left(S^i\mathfrak{g}^* \otimes \Omega^j(M)\right)$$

and

$$d_G(\alpha)(X) = d(\alpha(X)) - \imath_{\widetilde{X}}\alpha(X), \quad X \in \mathfrak{g}$$

where $\widetilde{X}_x := \frac{d}{dt}\Big|_{t=0} e^{-tX} \cdot x$ for all $x \in M$

Definition (Stienon '10, Meinrenken '03)

An object $\widehat{\mathcal{L}} = (L, \mu, \pi, \nabla, B) \in G\text{-Grb}_\nabla(M)$ consists of the following:

- A G-equivariant surjective submersion $Y \xrightarrow{\pi} M$.
- $(L, \nabla) \in G\text{-Bun}_{S^1,\nabla}(Y^{[2]})$ (Here $Y^{[2]} = Y \times_M Y$.)
- Isomorphism $\mu\colon \pi_{12}^*(L, \nabla) \otimes \pi_{23}^*(L, \nabla) \to \pi_{13}^*(L, \nabla)$ in $G\text{-Bun}_{S^1,\nabla}(Y^{[3]})$ that is associative over $Y^{[4]}$.
- $B \in \Omega_G^2(Y)$ satisfying $F_G(\nabla) = \pi_2^* B - \pi_1^* B \in \Omega_G^2(Y^{[2]})$.

Definition (P.-Redden, Waldorf)

For $\widehat{\mathcal{L}}_i = (L_i, \nabla_i, \mu_i, \pi_i, B_i) \in G\text{-Grb}_\nabla(M)$, an isomorphism $\widehat{\mathcal{L}}_1 \xrightarrow{\widehat{\mathcal{K}}} \widehat{\mathcal{L}}_2$ is a quadruple $(\zeta, K, \nabla_K, \alpha)$ consists of the following.

- a G-equivariant surjective submersion $\zeta\colon Z \to Y_1 \times_M Y_2$
- $(K, \nabla_K) \in G\text{-Bun}_{S^1,\nabla}(Z)$ such that $F_G(\nabla_K) = \zeta^*(B_2 - B_1) \in \Omega^2_G(Z)$.
- Isomorphism $\alpha\colon (L_1, \nabla_1) \otimes \zeta_2^*(K, \nabla_K) \to \zeta_1^*(K, \nabla_K) \otimes (L_2, \nabla_2)$ of G-equivariant S^1-bundles with connection over $Z \times_M Z$ compatible with μ_1 and μ_2.

Remark

(1) When $G = 1$ and $\zeta = \mathbf{1}$, we recover the stable isomorphism of Murray and Stevenson (2000).

(2) When $G = 1$, $G\text{-Bun}_{S^1,\nabla}(Z)$ is replaced by $\text{Bun}(Z)$, we recover the 1-morphism of Waldorf (2007).

Proposition (Waldorf, 2007)

There is an equivalence of groupoids between the 1-groupoid of 1-morphisms of $\text{Grb}(M)$ and the 1-groupoid of stable isomorphisms of $\text{Grb}_{\text{st}}(M)$.

Proposition (P.-Redden, 2017)

The above theorem of Waldorf holds for 1-morphisms between G-equivariant gerbes.

2-morphisms

Definition (P.-Redden, Waldorf)

A transformation $\widehat{\mathcal{J}}\colon \widehat{\mathcal{K}}_1 \Rightarrow \widehat{\mathcal{K}}_2$, which is a morphism in the groupoid $G\text{-Grb}_\nabla(\widehat{\mathcal{L}}_1, \widehat{\mathcal{L}}_2)$ and a 2-morphism in $G\text{-Grb}_\nabla(M)$, is an equivalence class of triples (W, ω, β_W) consists of the following:

(1) G-equivariant surjective submersion $\omega : W \to Z_1 \times_{Y_1 \times_M Y_2} Z_2$

(2) Isomorphism $\beta_W : (K_1, \nabla_1) \to (K_2, \nabla_2)$ over W compatible with α_1 and α_2.

$$\begin{array}{ccc} L_1 \otimes \omega_2^* K_1 & \xrightarrow{\alpha_1} & \omega_1^* K_1 \otimes L_2 \\ \Big\downarrow {\scriptstyle \mathbf{1} \otimes \omega_2^* \beta_W} & & \Big\downarrow {\scriptstyle \omega_1^* \beta_W \otimes \mathbf{1}} \\ L_1 \otimes \omega_2^* K_2 & \xrightarrow{\alpha_2} & \omega_1^* K_2 \otimes L_2 \end{array}$$

2-morphisms

Definition

Definition

$(W, \omega, \beta_W) \sim (W', \omega', \beta_{W'})$ if there is a G-manifold X with G-equivariant surjective submersions to W and W' such that the following diagram commutes

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow \omega \\ W' & \xrightarrow{\omega'} & Z_1 \times_{Y_1 \times_M Y_2} Z_2 \end{array}$$

and β_W and $\beta_{W'}$ coincides if pulled back to X.

Definition (Stiénon '10, Meinrenken '03)

Let $\widehat{\mathcal{L}} = (L, \mu, \pi, \nabla, B) \in G\text{-Grb}_{\nabla}(M)$. The **equivariant 3-curvature** of $\widehat{\mathcal{L}}$ is the equivariant 3-form $H_G \in \Omega^3_G(M)$ such that $\pi^* H_G = d_G B$.

Proposition (Stiénon '10)

(1) Equivariant connection ∇ and curving B always exists.

(2) The class $[H_G] \in H^3_G(M)$ is independent of the choice of connection ∇ and curving B.

The class $[H_G]$ is called **equivariant Diximier-Douady class** of $\widehat{\mathcal{L}}$ due to Stiénon and Meinrenken.

G-equivariant Dixímier-Douady Class

Consequences of Stiénon, Xu, Behrend, and Tu's work

Remark

(1) The main result of Stiénon (2010) is a comparison between the Stiénon-Meinrenken equivariant Dixmier-Douady class with Behrend-Xu equivariant Dixmier-Douady class.

(2) Consequence:

$$0 \to G\text{-Grb}(M)_{/\cong} \to H^3_G(M;\mathbb{Z}) \to 0.$$

(3) Surjectivity and $\mathbb{Z}$ coefficients are due to Tu and Xu (2015).

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Question

How about (2) above **"with connection"**?

Simplicial presheaves and ∞-stacks
Notations

$$\mathrm{Set} \hookrightarrow \mathrm{Gpd} \hookrightarrow 2\text{-}\mathrm{Gpd} \hookrightarrow \infty\text{-}\mathrm{Gpd}$$

Let $\mathcal{C}$ be an ∞-Gpd.

Definition

Let $Y \to M$ be a cover. The **Čech nerve** $Y^\bullet \in \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ is a simplicial object in $\mathcal{C}$

$$\begin{aligned} Y^\bullet &: \Delta^{\mathrm{op}} \to \mathcal{C} \\ [n] &\mapsto Y^\bullet([n]) = Y^{[n]} = \underbrace{Y \times_M \ldots \times_M Y}_{n} \end{aligned}$$

with obvious face and degeneracy maps from pullback squares and diagonal inclusions, respectively. i.e.

$$Y^\bullet := \ldots Y \times_M Y \times_M Y \overset{\longrightarrow}{\underset{\longrightarrow}{\overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}}}} Y \times_M Y \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} Y.$$

Simplicial presheaves and ∞-stacks

Prestacks and stacks

Definition

A **prestack** of $\mathcal{C}$ on the site of manifold **Man** is a functor $\mathcal{F} : \mathbf{Man}^{\mathrm{op}} \to \mathcal{C}$.

Definition

A **stack** $\mathcal{F}$ is a prestack that satisfies the descent condition; i.e. for any cover $Y \to M$,

$$\mathcal{F}(M) \xrightarrow{\cong} \mathrm{holim}_{\Delta} \mathcal{F}(Y^{\bullet}).$$

Remark

- If $\mathcal{C}$ is **Set**, then the descent condition is the usual sheaf condition.
- If $\mathcal{C}$ is **Gpd**, then the descent condition is the usual stack condition.

Notation

$\mathbf{Shv}_\infty$ the totality of stacks valued in $\mathcal{C} = \mathbf{Grpd}_\infty$

Example

- $M \in \mathbf{Shv}_\infty$: $M(X) := C^\infty(X, M) \in \mathbf{Set}$
- $\Omega^k \in \mathbf{Shv}_\infty$: $\Omega^k(X) \in \mathbf{Set}$
- $\mathcal{B}G \in \mathbf{Shv}_\infty$: $\mathcal{B}G(X) := \mathbf{Bun}_G(X) \in \mathbf{Grpd}$
- $\mathcal{B}_\nabla G \in \mathbf{Shv}_\infty$: $\mathcal{B}_\nabla G(X) := \mathbf{Bun}_{\nabla, G}(X) \in \mathbf{Grpd}$
- $\mathcal{B}^2_\nabla S^1 \in \mathbf{Shv}_\infty$:

$$\mathcal{B}^2_\nabla S^1(X) := L(\Gamma(S^1 \xrightarrow{d\log} \Omega^1 \to \Omega^2))(X) \in \mathbf{Grpd}_\infty$$

- $\mathcal{B}^{p+1}_\nabla S^1 \in \mathbf{Shv}_\infty$:

$$\mathcal{B}^{p+1}_\nabla S^1(X) := L(\Gamma(S^1 \xrightarrow{d\log} \Omega^1 \to \Omega^2 \to \cdots \to \Omega^{p+1}))(X) \in \mathbf{Grpd}_\infty.$$

Simplicial presheaves and ∞-stacks

Example: Differential Quotient Stack

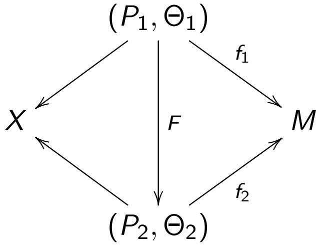
$$
\begin{aligned}
\mathcal{E}_\nabla G \times_G M : \mathbf{Man}^{\mathrm{op}} &\to \mathbf{Grpd} \\
X &\mapsto \mathcal{E}_\nabla G \times_G M(X) \in \mathbf{Grpd}
\end{aligned}
$$

- Objects:

$$X \leftarrow (P, \Theta) \xrightarrow{f} M$$

where $(P, \Theta) \in \mathbf{Bun}_{G,\nabla}(X)$, $M \in G$-$\mathbf{Man}$, and $f \in G$-$\mathbf{Man}(P, M)$.

- Morphisms:



where $F^*\Theta_2 = \Theta_1^*$ and $f_2 \circ F = f_1$.

Definition

(1) A stack $\mathcal{F} \in \mathbf{Shv}_\infty$ is called **homotopy invariant** if $\mathcal{F}(M \times I) \cong \mathcal{F}(M)$.

(2) A **homotopification** $\mathcal{H}$ is a left-adjoint of inclusion.

$$\mathbf{Shv}_\infty \underset{\text{incl}}{\overset{\mathcal{H}}{\rightleftarrows}} \mathbf{Shv}_\infty^h$$

(3) For any $Y \in \mathbf{Top}$, $\mathrm{Sing}_\bullet(Y) \in \mathbf{Shv}_\infty$ defined by $\mathrm{Sing}_\bullet(Y)(X) := \mathrm{Sing}_\bullet \mathbf{Top}(X, Y)$.

Notation

$$h : \mathbf{Shv}_\infty \underset{\text{incl}}{\overset{\mathcal{H}}{\rightleftarrows}} \mathbf{Shv}_\infty^h \underset{\mathrm{Sing}_\bullet}{\overset{|\mathrm{ev}(\mathrm{pt})|}{\rightleftarrows}} \mathbf{Top} : \mathrm{Sing}_\bullet$$

Definition

$\mathcal{K}(A, n) \in \mathbf{Shv}_\infty$ is a simplicial sheaf homotopy equivalent to $\mathrm{Sing}_\bullet(K(A, n))$ for $A \in \mathbf{Ab}$.

Proposition

The pullback object $\widehat{\mathcal{K}}(\mathbb{Z}, n) \in \mathbf{Shv}_\infty$ in the diagram

$$\begin{array}{ccc} \widehat{\mathcal{K}}(\mathbb{Z}, n) & \longrightarrow & \Omega^n_{\text{closed}} \\ \downarrow & & \downarrow \\ \mathcal{K}(\mathbb{Z}, n) & \longrightarrow & \mathcal{K}(\mathbb{R}, n) \end{array}$$

is equivalent to $(n-2)$-Grb_∇ in $\mathbf{Shv}_\infty$.

How do we study equivariant theories?

Idea

Non-equivariant case

Let $M \in \mathbf{Man}$ and $\mathcal{F} \in \mathbf{Shv}_\infty$.

$$\varphi \in \mathcal{F}(M) \qquad \longleftrightarrow \qquad M \xrightarrow{\varphi} \mathcal{F} \quad \in \mathbf{Shv}_\infty(M, \mathcal{F})$$

Equivariant case

Let $M \in G\text{-}\mathbf{Man}$ and $\mathcal{E}_M, \mathcal{F} \in \mathbf{Shv}_\infty$.

$$\psi \in \mathcal{F}_G(M) \qquad \longleftrightarrow \qquad \mathcal{E}_M \xrightarrow{\psi} \mathcal{F} \quad \in \mathbf{Shv}_\infty(\mathcal{E}_M, \mathcal{F})$$

Example

(1) $H^n_G(M;A) \quad \overset{\text{``}=\text{''}}{\longleftrightarrow} \quad H^n(\mathcal{E}_\nabla G \times_G M; A)$

Fact (Redden, 2016): $EG \times_G M \cong \mathsf{h}(\mathcal{E}G \times_G M) \cong \mathsf{h}(\mathcal{E}_\nabla G \times_G M)$

$$\begin{aligned} H^n(\mathcal{E}_\nabla G \times_G M; A) :=& \mathrm{ho}\mathbf{Shv}_\infty(\mathcal{E}_\nabla G \times_G M, \mathrm{Sing}_\bullet(K(A,n))) \\ \cong& \mathrm{ho}\mathbf{Top}(\mathsf{h}(\mathcal{E}_\nabla G \times_G M), K(A,n)) \\ \cong& \pi_0 \mathbf{Top}(EG \times_G M, K(A,n)) =: H^n_G(M) \end{aligned}$$

How do we study equivariant theories?

Examples

Example (Freed-Hopkins, 2013)

(2)

$$\Omega^n_G(M) \quad \stackrel{\cong}{\longleftrightarrow} \quad \Omega^n(\mathcal{E}_\nabla G \times_G M).$$

The map is given by the Weil homomorphism. Given $\alpha \wedge \beta \wedge \gamma \in \Omega_G(M) = (S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes \Omega(M))_{\text{basic}}$ with $|S^1\mathfrak{g}^*| = 2$ and $|\Lambda^1\mathfrak{g}^*| = 1$,

$$\begin{array}{ccc} (P, \Theta) & \xrightarrow{f} & M \\ \downarrow & & \\ X & & \end{array}$$

$$\Omega_G(M) \xrightarrow{f^*} \Omega_G(P) \xrightarrow{\Theta^*} \Omega(P)_{\text{basic}} \cong \Omega(X)$$

$$\alpha \wedge \beta \wedge \gamma \mapsto f^*(\alpha \wedge \beta \wedge \gamma) \mapsto f^*\alpha(F_\Theta) \wedge f^*\beta(\Theta) \wedge f^*\gamma$$

$$X \xrightarrow{(P,\Theta,f)} \mathcal{E}_\nabla G \times_G M \qquad \Omega(\mathcal{E}_\nabla G \times_G M) \xrightarrow{(P,\Theta,f)^*} \Omega(X)$$

Definition

$$\widehat{H}^n_G(M) := \widehat{H}^n(\mathcal{E}_\nabla G \times_G M) := \mathrm{ho}\mathbf{Shv}_\infty(\mathcal{E}_\nabla G \times_G M, \widehat{H}^n)$$

In other words, $\widehat{\lambda} \in \widehat{H}^n_G(M)$ is a construction

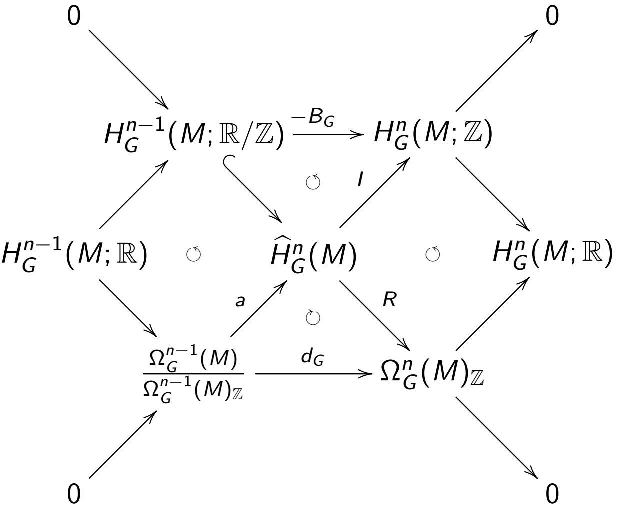
- $$X \leftarrow (P, \Theta) \xrightarrow{f} M \rightsquigarrow \widehat{\lambda}(P, \Theta, f) \in \widehat{H}^n(X)$$
- If

$$\begin{array}{ccccc} (P_1, \Theta_1) & \xrightarrow{\widetilde{\varphi}} & (P_2, \Theta_2) & \xrightarrow{f} & M \\ \downarrow & & \downarrow & & \\ X_1 & \longrightarrow & X_2 & & \end{array}$$

then $\varphi^* \widehat{\lambda}(P_2, \Theta_2, f) = \widehat{\lambda}(P_1, \Theta_1, f \circ \widetilde{\varphi})$.

Equivariant Differential Cohomology

Hexagon diagram à la Redden



Ordinary Differential Cohomology

Cochain Model

$$\begin{array}{ccc}
\widehat{C}^n(M) & \longrightarrow & \Omega^n(M;\mathbb{R}) \\
\downarrow & & \downarrow \\
C^n(M;\mathbb{Z}) & \longrightarrow & C^n(M;\mathbb{R})
\end{array}$$

- $\widehat{C}^n(M) = C^n(M;\mathbb{Z}) \times C^{n-1}(M;\mathbb{R}) \times \Omega^n(M;\mathbb{R})$
- $\widehat{d}(c, h, \omega) = (\delta c, c - \int \omega - \delta h, d\omega)$
- $\widehat{H}^\bullet(M) := \mathrm{Ker}(\widehat{d})/\mathrm{Im}(\widehat{d})$.

Equivariant Differential Cohomology

Cochain Model

$$\begin{array}{ccc} \widehat{C}^n_G(M) & \longrightarrow & \Omega^n_G(M;\mathbb{R}) \\ \downarrow & & \downarrow \\ C^n_G(M;\mathbb{Z}) & \longrightarrow & C^n_G(M;\mathbb{R}) \end{array}$$

- $\widehat{C}^n_G(M) = C^n_G(M;\mathbb{Z}) \times C^{n-1}_G(M;\mathbb{R}) \times \Omega^n_G(M;\mathbb{R})$
- $\widehat{d}_G(c, h, \omega) = (\delta c, c - \int \omega - \delta h, d_G\omega)$
- $\widehat{H}^\bullet_G(M) :=$ cohomology of $(\widehat{C}^\bullet_G(M), \widehat{d}_G)$.

Theorem (Redden, 2016)

The two definitions of $\widehat{H}^\bullet_G(M)$ are naturally isomorphic:

$$\mathrm{ho}\mathbf{Shv}_\infty(\mathcal{E}_\nabla G \times_G M, \widehat{H}^\bullet) \cong \widehat{H}^\bullet_G(M).$$

Main Theorems

Theorem (P. and Redden)

Let $M \in G$-Man, with G a compact Lie group.

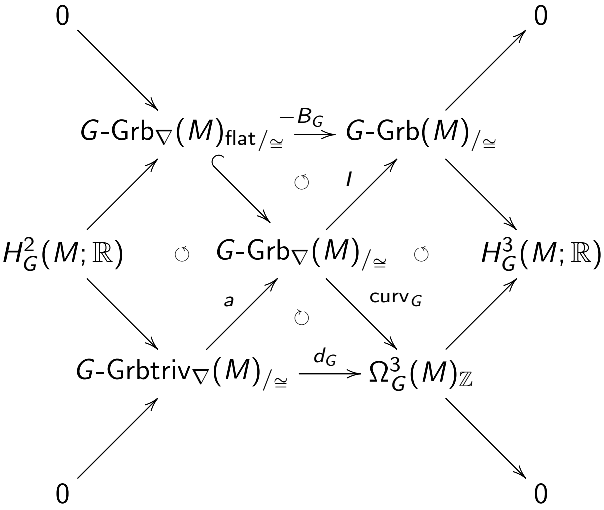
(1) There is a natural isomorphism of abelian groups

$$G\text{-Grb}_{\nabla}(M)_{/\cong} \xrightarrow{\cong} \widehat{H}^3_G(M).$$

(2) Furthermore, Redden's equivariant differential cohomology hexagon diagram is naturally isomorphic to ...

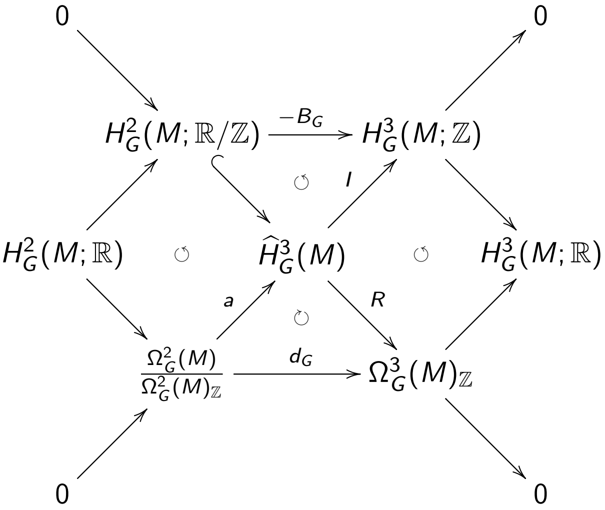
Main Theorems

Redden's hexagon diagram in degree 3 with geometric cocycles



Main Theorems

Hexagon diagram à la Redden in degree 3



Main Theorems

Theorem (P. and Redden)

There exists a natural functor

$$G\text{-}\mathsf{Grb}_{\nabla}(M) \longrightarrow \mathsf{Grb}_{\nabla}(\mathcal{E}_{\nabla}G \times_G M)$$

which is an equivalence of 2-groupoids.

Thank you!