

A classification of equivariant gerbe connections

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G -equivariant bundle gerbes with connection

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Equivariant differential cohomology

Main theorems

Differential Cohomology

Hopkins-Singer Cochain Model

$$\begin{array}{ccc} \widehat{C}^n(M) & \longrightarrow & \Omega^n(M; \mathbb{R}) \\ \downarrow & & \downarrow \\ C^n(M; \mathbb{Z}) & \longrightarrow & C^n(M; \mathbb{R}) \end{array}$$

- $\widehat{C}^n(M) = C^n(M; \mathbb{Z}) \times C^{n-1}(M; \mathbb{R}) \times \Omega^n(M; \mathbb{R})$
- $\widehat{d}(c, h, \omega) = (\delta c, c - \int \omega - \delta h, d\omega)$
- $\widehat{H}^\bullet(M) := \text{Ker}(\widehat{d}) / \text{Im}(\widehat{d})$.

Differential Cohomology

Hexagon diagram à la Cheeger and Simons

The diagram is a hexagon with vertices and arrows as follows:

- Top-left vertex: 0
- Top-right vertex: 0
- Middle-left vertex: $H^{n-1}(M; \mathbb{R}/\mathbb{Z})$
- Middle-right vertex: $H^n(M; \mathbb{Z})$
- Bottom-left vertex: $H^{n-1}(M; \mathbb{R})$
- Bottom-right vertex: $H^n(M; \mathbb{R})$
- Inner-left vertex: $\frac{\Omega^{n-1}(M)}{\Omega^{n-1}(M)_{\mathbb{Z}}}$
- Inner-right vertex: $\Omega^n(M)_{\mathbb{Z}}$
- Center vertex: $\widehat{H}^n(M)$

Arrows and maps:

- $0 \rightarrow H^{n-1}(M; \mathbb{R}/\mathbb{Z})$ (diagonal down-left)
- $0 \rightarrow H^n(M; \mathbb{Z})$ (diagonal down-right)
- $H^{n-1}(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{-B} H^n(M; \mathbb{Z})$ (horizontal top)
- $H^{n-1}(M; \mathbb{R}) \rightarrow H^{n-1}(M; \mathbb{R}/\mathbb{Z})$ (diagonal up-left)
- $H^n(M; \mathbb{Z}) \rightarrow H^n(M; \mathbb{R})$ (diagonal down-right)
- $H^{n-1}(M; \mathbb{R}) \rightarrow \frac{\Omega^{n-1}(M)}{\Omega^{n-1}(M)_{\mathbb{Z}}}$ (diagonal down-left)
- $H^n(M; \mathbb{R}) \rightarrow \Omega^n(M)_{\mathbb{Z}}$ (diagonal down-right)
- $\frac{\Omega^{n-1}(M)}{\Omega^{n-1}(M)_{\mathbb{Z}}} \xrightarrow{d} \Omega^n(M)_{\mathbb{Z}}$ (horizontal bottom)
- $\widehat{H}^n(M) \rightarrow H^{n-1}(M; \mathbb{R}/\mathbb{Z})$ (diagonal up-left)
- $\widehat{H}^n(M) \rightarrow H^n(M; \mathbb{Z})$ (diagonal up-right)
- $\widehat{H}^n(M) \rightarrow \frac{\Omega^{n-1}(M)}{\Omega^{n-1}(M)_{\mathbb{Z}}}$ (diagonal down-left, labeled a)
- $\widehat{H}^n(M) \rightarrow \Omega^n(M)_{\mathbb{Z}}$ (diagonal down-right, labeled R)
- $\widehat{H}^n(M) \rightarrow H^{n-1}(M; \mathbb{R}/\mathbb{Z})$ (curved arrow, labeled \circlearrowleft)
- $\widehat{H}^n(M) \rightarrow H^n(M; \mathbb{Z})$ (curved arrow, labeled \circlearrowleft)
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- $\widehat{H}^n(M) \rightarrow H^n(M; \mathbb{R})$ (curved arrow, labeled \circlearrowleft)
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What is a gerbe?

Geometric way to represent elements of $H^2(M; \mathbb{Z})$

M : a smooth manifold.

There are geometric cocycles representing elements of $H^2(M; \mathbb{Z})$.

(L, ∇) a complex line bundle over M with connection ∇ .

$c_1(\nabla) := \frac{i}{2\pi} F_\nabla$: the first *Chern class* of ∇ .

Chern-Weil theorem says $c_1(\nabla) \in \Omega^2(M; \mathbb{C})$ is a topological invariant of a line bundle L . i.e.

$c_1(\nabla) - c_1(\nabla') = d_{\text{CS}}(t \mapsto ((1-t)\nabla + t\nabla'))$.

$c_1(L) \in H_{\text{dR}}^2(M; \mathbb{C})$ and in fact $c_1(L) \in H^2(M; \mathbb{Z})$

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Question

How do we geometrically represent elements of $H^n(M; \mathbb{Z})$ for $n = 3, 4, \dots$? What is the higher analogue of line bundles?

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Answer: $(n - 2)$ -gerbes

What is a gerbe?

Hierarchy of $(n - 2)$ -gerbes with connection and differential cohomology

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \uparrow & & \uparrow & & \uparrow & \\ & \delta & & \delta & & \delta & \\ C^2(\mathcal{U}, \underline{S^1}) & \xrightarrow{d} & C^2(\mathcal{U}, \Omega_M^1) & \xrightarrow{d} & C^2(\mathcal{U}, \Omega_M^2) & \xrightarrow{d} & \dots \\ & \uparrow & & \uparrow & & \uparrow & \\ C^1(\mathcal{U}, \underline{S^1}) & \xrightarrow{d} & C^1(\mathcal{U}, \Omega_M^1) & \xrightarrow{d} & C^1(\mathcal{U}, \Omega_M^2) & \xrightarrow{d} & \dots \\ & \uparrow & & \uparrow & & \uparrow & \\ C^0(\mathcal{U}, \underline{S^1}) & \xrightarrow{d} & C^0(\mathcal{U}, \Omega_M^1) & \xrightarrow{d} & C^0(\mathcal{U}, \Omega_M^2) & \xrightarrow{d} & \dots \end{array}$$

$$\widehat{H}^1(M) \cong C^\infty(M, S^1)$$

$$\widehat{H}^2(M) \cong \text{Bun}_{S^1, \nabla}(M) / \cong$$

$$\widehat{H}^3(M) \cong \text{Grb}_{\nabla}(M) / \cong$$

What is a gerbe?

Example: An obstruction for lifting a principal bundle

$$1 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \text{ a central extension of } G.$$

Question

Let $P \rightarrow M$ be a principal G -bundle. What is the necessary and sufficient condition for lifting a principal G -bundle P to a principal \tilde{G} -bundle \tilde{P} ?

What is a gerbe?

Example: An obstruction for lifting a principal bundle

$1 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ a central extension of G .

Question

Let $P \rightarrow M$ be a principal G -bundle. What is the necessary and sufficient condition for lifting a principal G -bundle P to a principal \tilde{G} -bundle \tilde{P} ?

Answer: Choose a good cover $\mathcal{U} = \{U_i\}$ of M . The transition maps $g_{ij} : U_{ij} \rightarrow G$ are lifted to $\tilde{g}_{ij} : U_{ij} \rightarrow \tilde{G}$. Define $\lambda_{ijk} := \tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki}$. So P lifts to \tilde{P} if and only if

- λ_{ijk} represents $1 \in \check{H}^2(\mathcal{U}, A)$.
- The gerbe with cocycle λ is trivializable.

S^1 -bundle gerbes with connection

Definition, characteristic class

Definition (Murray '94)

Let $M \in \mathbf{Man}$. An object $\mathcal{L} = (L, \mu, \pi) \in \mathbf{Grb}(M)$ consists of the following:

- A surjective submersion $Y \xrightarrow{\pi} M$.
- $L \in \mathbf{Bun}_{S^1}(Y^{[2]})$. (Here $Y^{[2]} = Y \times_M Y$.)
- Isomorphism $\mu: \pi_{12}^* L \otimes \pi_{23}^* L \rightarrow \pi_{13}^* L$ in $\mathbf{Bun}_{S^1}(Y^{[3]})$ that is associative over $Y^{[4]}$.

S^1 -bundle gerbes with connection

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After choosing an open cover $\mathcal{U} = \{U_i\}$, the isomorphism μ determines a degree 2 Čech cochain $g_{ijk}: U_{ijk} \rightarrow S^1$. The associativity of μ is the condition for g being a cocycle. Hence we obtain a class in $H^3(M; \mathbb{Z})$ which is called the **Dixmier-Douady class** of \mathcal{L}

S^1 -bundle gerbes with connection

Definition

Definition (Murray '94)

Let $M \in \mathbf{Man}$. An object $\widehat{\mathcal{L}} = (L, \mu, \pi, \nabla, B) \in \text{Grb}_{\nabla}(M)$ consists of the following:

- A surjective submersion $Y \xrightarrow{\pi} M$.
- $(L, \nabla) \in \text{Bun}_{S^1, \nabla}(Y^{[2]})$. (Here $Y^{[2]} = Y \times_M Y$.)
- Isomorphism $\mu: \pi_{12}^*(L, \nabla) \otimes \pi_{23}^*(L, \nabla) \rightarrow \pi_{13}^*(L, \nabla)$ in $\text{Bun}_{S^1, \nabla}(Y^{[3]})$ that is associative over $Y^{[4]}$.
- $B \in \Omega^2(Y)$ satisfying $F_{\nabla} = \pi_2^* B - \pi_1^* B =: \delta B \in \Omega^2(Y^{[2]})$.

S^1 -bundle gerbes with connection

The 3-curvature

- Since $0 = dF_{\nabla} = d\delta B = \delta dB$ and the sequence

$$0 \rightarrow \Omega^3(M) \xrightarrow{\pi^*} \Omega^3(Y) \xrightarrow{\delta} \Omega^3(Y^{[2]}) \xrightarrow{\delta} \dots$$

is an acyclic complex, there exists a unique $H \in \Omega_{\text{closed}}^3(M)$ such that $\pi^* H = dB$. This closed 3-form is called the **3-curvature** of bundle gerbe.

- The de Rham cohomology class $[H/2\pi i] \in H_{\text{dR}}^3(M; \mathbb{R})$ is in fact an integral cohomology class equal to the Dixmier-Douady class of $\mathcal{L} = (L, \mu, \pi)$.

Interlude: Equivariant cohomology and Cartan model of G -equivariant de Rham complex

From now on, G denotes a compact Lie group acting on a smooth manifold M .

Definition (Borel equivariant cohomology)

Let A be any abelian group. $H_G^\bullet(M; A) := H^\bullet(EG \times_G M; A)$.

Definition (Cartan model)

The G -equivariant de Rham complex is a complex (Ω_G^\bullet, d_G) where

$$\Omega_G^k(M) := \bigoplus_{k=2i+j} (S^i \mathfrak{g}^* \otimes \Omega^j(M))$$

and

$$d_G(\alpha)(X) = d(\alpha(X)) - \iota_{\tilde{X}}\alpha(X), \quad X \in \mathfrak{g}$$

where $\tilde{X}_x := \left. \frac{d}{dt} \right|_{t=0} e^{-tX} \cdot x$ for all $x \in M$

G -equivariant S^1 -bundle gerbes with connection

Definition

Definition (Stienon '10, Meinrenken '03)

An object $\widehat{\mathcal{L}} = (L, \mu, \pi, \nabla, B) \in G\text{-Grb}_{\nabla}(M)$ consists of the following:

- A G -equivariant surjective submersion $Y \xrightarrow{\pi} M$.
- $(L, \nabla) \in G\text{-Bun}_{S^1, \nabla}(Y^{[2]})$ (Here $Y^{[2]} = Y \times_M Y$.)
- Isomorphism $\mu: \pi_{12}^*(L, \nabla) \otimes \pi_{23}^*(L, \nabla) \rightarrow \pi_{13}^*(L, \nabla)$ in $G\text{-Bun}_{S^1, \nabla}(Y^{[3]})$ that is associative over $Y^{[4]}$.
- $B \in \Omega_G^2(Y)$ satisfying $F_G(\nabla) = \pi_2^* B - \pi_1^* B \in \Omega_G^2(Y^{[2]})$.

1-morphisms

Definition

Definition (P.-Redden, Waldorf)

For $\widehat{\mathcal{L}}_i = (L_i, \nabla_i, \mu_i, \pi_i, B_i) \in G\text{-Grb}_{\nabla}(M)$, an isomorphism $\widehat{\mathcal{L}}_1 \xrightarrow{\widehat{\mathcal{K}}} \widehat{\mathcal{L}}_2$ is a quadruple $(\zeta, K, \nabla_K, \alpha)$ consists of the following.

- a G -equivariant surjective submersion $\zeta: Z \rightarrow Y_1 \times_M Y_2$
- $(K, \nabla_K) \in G\text{-Bun}_{S^1, \nabla}(Z)$ such that $F_G(\nabla_K) = \zeta^*(B_2 - B_1) \in \Omega_G^2(Z)$.
- Isomorphism $\alpha: (L_1, \nabla_1) \otimes \zeta_2^*(K, \nabla_K) \rightarrow \zeta_1^*(K, \nabla_K) \otimes (L_2, \nabla_2)$ of G -equivariant S^1 -bundles with connection over $Z \times_M Z$ compatible with μ_1 and μ_2 .

1-morphisms

1-morphism vs. stable isomorphism

Remark

- (1) When $G = 1$ and $\zeta = \mathbf{1}$, we recover the [stable isomorphism](#) of Murray and Stevenson (2000).
- (2) When $G = 1$, $G\text{-Bun}_{S^1, \nabla}(Z)$ is replaced by $\text{Bun}(Z)$, we recover the 1-morphism of Waldorf (2007).

Proposition (Waldorf, 2007)

There is an equivalence of groupoids between the 1-groupoid of 1-morphisms of $\text{Grb}(M)$ and the 1-groupoid of stable isomorphisms of $\text{Grb}_{\text{st}}(M)$.

Proposition (P.-Redden, 2017)

The above theorem of Waldorf holds for 1-morphisms between G -equivariant gerbes.

2-morphisms

Definition

Definition (P.-Redden, Waldorf)

A transformation $\widehat{\mathcal{J}}: \widehat{\mathcal{K}}_1 \Rightarrow \widehat{\mathcal{K}}_2$, which is a morphism in the groupoid $G\text{-Grb}_\nabla(\widehat{\mathcal{L}}_1, \widehat{\mathcal{L}}_2)$ and a 2-morphism in $G\text{-Grb}_\nabla(M)$, is an equivalence class of triples (W, ω, β_W) consists of the following:

- (1) G -equivariant surjective submersion $\omega: W \rightarrow Z_1 \times_{Y_1 \times_M Y_2} Z_2$
- (2) Isomorphism $\beta_W: (K_1, \nabla_1) \rightarrow (K_2, \nabla_2)$ over W compatible with α_1 and α_2 .

$$\begin{array}{ccc} L_1 \otimes \omega_2^* K_1 & \xrightarrow{\alpha_1} & \omega_1^* K_1 \otimes L_2 \\ \downarrow \mathbf{1} \otimes \omega_2^* \beta_W & & \downarrow \omega_1^* \beta_W \otimes \mathbf{1} \\ L_1 \otimes \omega_2^* K_2 & \xrightarrow{\alpha_2} & \omega_1^* K_2 \otimes L_2 \end{array}$$

2-morphisms

Definition

Definition

$(W, \omega, \beta_W) \sim (W', \omega', \beta_{W'})$ if there is a G -manifold X with G -equivariant surjective submersions to W and W' such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad} & W \\ \downarrow & & \downarrow \omega \\ W' & \xrightarrow{\omega'} & Z_1 \times_{Y_1 \times_M Y_2} Z_2 \end{array}$$

and β_W and $\beta_{W'}$ coincides if pulled back to X .

G -equivariant Diximier-Douady Class

Definition

Definition (Stiénon '10, Meinrenken '03)

Let $\widehat{\mathcal{L}} = (L, \mu, \pi, \nabla, B) \in G\text{-Grb}_{\nabla}(M)$. The **equivariant 3-curvature** of $\widehat{\mathcal{L}}$ is the equivariant 3-form $H_G \in \Omega_G^3(M)$ such that $\pi^* H_G = d_G B$.

Proposition (Stiénon '10)

- (1) Equivariant connection ∇ and curving B always exists.
- (2) The class $[H_G] \in H_G^3(M)$ is independent of the choice of connection ∇ and curving B .

The class $[H_G]$ is called **equivariant Diximier-Douady class** of $\widehat{\mathcal{L}}$ due to Stiénon and Meinrenken.

G -equivariant Dixmier-Douady Class

Consequences of Stiénon, Xu, Behrend, and Tu's work

Remark

- (1) The main result of Stiénon (2010) is a comparison between the Stiénon-Meinrenken equivariant Dixmier-Douady class with Behrend-Xu equivariant Dixmier-Douady class.
- (2) Consequence:

$$0 \rightarrow G\text{-Grb}(M)_{/\cong} \rightarrow H_G^3(M; \mathbb{Z}) \rightarrow 0.$$

- (3) Surjectivity and \mathbb{Z} coefficients are due to Tu and Xu (2015).

G-equivariant Diximier-Douady Class

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Question

How about (2) above **“with connection”**?

Simplicial presheaves and ∞ -stacks

Notations

$$\text{Set} \hookrightarrow \text{Gpd} \hookrightarrow 2\text{-Gpd} \hookrightarrow \infty\text{-Gpd}$$

Let \mathcal{C} be an ∞ -Gpd.

Definition

Let $Y \rightarrow M$ be a cover. The **Čech nerve** $Y^\bullet \in \mathbf{Fun}(\Delta^{\text{op}}, \mathcal{C})$ is a simplicial object in \mathcal{C}

$$Y^\bullet : \Delta^{\text{op}} \rightarrow \mathcal{C}$$

$$[n] \mapsto Y^\bullet([n]) = Y^{[n]} = \underbrace{Y \times_M \dots \times_M Y}_n$$

with obvious face and degeneracy maps from pullback squares and diagonal inclusions, respectively. i.e.

$$Y^\bullet := \dots Y \times_M Y \times_M Y \begin{matrix} \xrightarrow{\quad} \\ \xleftrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{matrix} Y \times_M Y \begin{matrix} \xrightarrow{\quad} \\ \xleftrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{matrix} Y.$$

Simplicial presheaves and ∞ -stacks

Prestacks and stacks

Definition

A **prestack** of \mathcal{C} on the site of manifold **Man** is a functor $\mathcal{F} : \mathbf{Man}^{\text{op}} \rightarrow \mathcal{C}$.

Definition

A **stack** \mathcal{F} is a prestack that satisfies the descent condition; i.e. for any cover $Y \rightarrow M$,

$$\mathcal{F}(M) \xrightarrow{\cong} \text{holim}_{\Delta} \mathcal{F}(Y^{\bullet}).$$

Remark

- If \mathcal{C} is **Set**, then the descent condition is the usual **sheaf** condition.
- If \mathcal{C} is **Gpd**, then the descent condition is the usual **stack** condition.

Simplicial presheaves and ∞ -stacks

Examples of stacks

Notation

\mathbf{Shv}_∞ the totality of stacks valued in $\mathcal{C} = \mathbf{Grpd}_\infty$

Example

- $M \in \mathbf{Shv}_\infty$: $M(X) := C^\infty(X, M) \in \mathbf{Set}$
- $\Omega^k \in \mathbf{Shv}_\infty$: $\Omega^k(X) \in \mathbf{Set}$
- $BG \in \mathbf{Shv}_\infty$: $BG(X) := \mathbf{Bun}_G(X) \in \mathbf{Grpd}$
- $B_{\nabla}G \in \mathbf{Shv}_\infty$: $B_{\nabla}G(X) := \mathbf{Bun}_{\nabla, G}(X) \in \mathbf{Grpd}$
- $B_{\nabla}^2 S^1 \in \mathbf{Shv}_\infty$:

$$B_{\nabla}^2 S^1(X) := L(\Gamma(S^1 \xrightarrow{d \log} \Omega^1 \rightarrow \Omega^2))(X) \in \mathbf{Grpd}_\infty$$

- $B_{\nabla}^{p+1} S^1 \in \mathbf{Shv}_\infty$:

$$B_{\nabla}^{p+1} S^1(X) := L(\Gamma(S^1 \xrightarrow{d \log} \Omega^1 \rightarrow \Omega^2 \rightarrow \dots \rightarrow \Omega^{p+1}))(X) \in \mathbf{Grpd}_\infty$$

Simplicial presheaves and ∞ -stacks

Example: Differential Quotient Stack

$$\mathcal{E}_{\nabla}G \times_G M : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Grpd}$$

$$X \mapsto \mathcal{E}_{\nabla}G \times_G M(X) \in \mathbf{Grpd}$$

- **Objects:**

$$X \leftarrow (P, \Theta) \xrightarrow{f} M$$

where $(P, \Theta) \in \mathbf{Bun}_{G, \nabla}(X)$, $M \in G\text{-Man}$, and $f \in G\text{-Man}(P, M)$.

- **Morphisms:**

$$\begin{array}{ccc} & (P_1, \Theta_1) & \\ & \swarrow & \searrow^{f_1} \\ X & & M \\ & \nwarrow & \nearrow_{f_2} \\ & (P_2, \Theta_2) & \\ & \downarrow F & \end{array}$$

where $F^*\Theta_2 = \Theta_1^*$ and $f_2 \circ F = f_1$.

Simplicial presheaves and ∞ -stacks

Some adjunctions

Definition

- (1) A stack $\mathcal{F} \in \mathbf{Shv}_\infty$ is called **homotopy invariant** if $\mathcal{F}(M \times I) \cong \mathcal{F}(M)$.
- (2) A **homotopification** \mathcal{H} is a left-adjoint of inclusion.

$$\mathbf{Shv}_\infty \begin{array}{c} \xrightarrow{\mathcal{H}} \\ \xleftarrow{\text{incl}} \end{array} \mathbf{Shv}_\infty^h$$

- (3) For any $Y \in \mathbf{Top}$, $\mathbf{Sing}_\bullet(Y) \in \mathbf{Shv}_\infty$ defined by $\mathbf{Sing}_\bullet(Y)(X) := \mathbf{Sing}_\bullet \mathbf{Top}(X, Y)$.

Notation

$$h : \mathbf{Shv}_\infty \begin{array}{c} \xrightarrow{\mathcal{H}} \\ \xleftarrow{\text{incl}} \end{array} \mathbf{Shv}_\infty^h \begin{array}{c} | \text{ev}(\text{pt}) | \\ \xleftarrow{\text{Sing}_\bullet} \end{array} \mathbf{Top} : \mathbf{Sing}_\bullet$$

Simplicial presheaves and ∞ -stacks

Gerbe connections as simplicial sheaves

Definition

$\mathcal{K}(A, n) \in \mathbf{Shv}_\infty$ is a simplicial sheaf homotopy equivalent to $\mathrm{Sing}_\bullet(K(A, n))$ for $A \in \mathbf{Ab}$.

Proposition

The pullback object $\widehat{\mathcal{K}}(\mathbb{Z}, n) \in \mathbf{Shv}_\infty$ in the diagram

$$\begin{array}{ccc} \widehat{\mathcal{K}}(\mathbb{Z}, n) & \longrightarrow & \Omega_{\mathrm{closed}}^n \\ \downarrow & & \downarrow \\ \mathcal{K}(\mathbb{Z}, n) & \longrightarrow & \mathcal{K}(\mathbb{R}, n) \end{array}$$

is equivalent to $(n - 2)\text{-Grb}_\nabla$ in \mathbf{Shv}_∞ .

How do we study equivariant theories?

Idea

Non-equivariant case

Let $M \in \mathbf{Man}$ and $\mathcal{F} \in \mathbf{Shv}_\infty$.

$$\varphi \in \mathcal{F}(M) \quad \longleftrightarrow \quad M \xrightarrow{\varphi} \mathcal{F} \in \mathbf{Shv}_\infty(M, \mathcal{F})$$

Equivariant case

Let $M \in G\text{-}\mathbf{Man}$ and $\mathcal{E}_M, \mathcal{F} \in \mathbf{Shv}_\infty$.

$$\psi \in \mathcal{F}_G(M) \quad \longleftrightarrow \quad \mathcal{E}_M \xrightarrow{\psi} \mathcal{F} \in \mathbf{Shv}_\infty(\mathcal{E}_M, \mathcal{F})$$

How do we study equivariant theories?

Examples

Example

$$(1) H_G^n(M; A) \xleftrightarrow{“=”} H^n(\mathcal{E}_{\nabla}G \times_G M; A)$$

Fact (Redden, 2016): $EG \times_G M \cong h(\mathcal{E}G \times_G M) \cong h(\mathcal{E}_{\nabla}G \times_G M)$

$$\begin{aligned} H^n(\mathcal{E}_{\nabla}G \times_G M; A) &:= \text{ho}\mathbf{Shv}_{\infty}(\mathcal{E}_{\nabla}G \times_G M, \text{Sing}_{\bullet}(K(A, n))) \\ &\cong \text{ho}\mathbf{Top}(h(\mathcal{E}_{\nabla}G \times_G M), K(A, n)) \\ &\cong \pi_0 \mathbf{Top}(EG \times_G M, K(A, n)) =: H_G^n(M) \end{aligned}$$

How do we study equivariant theories?

Examples

Example (Freed-Hopkins, 2013)

(2)

$$\Omega_G^n(M) \xleftarrow{\cong} \Omega^n(\mathcal{E}_{\nabla}G \times_G M).$$

The map is given by the Weil homomorphism. Given

$\alpha \wedge \beta \wedge \gamma \in \Omega_G(M) = (S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes \Omega(M))_{\text{basic}}$ with $|S^1\mathfrak{g}^*| = 2$ and $|\Lambda^1\mathfrak{g}^*| = 1$,

$$\begin{array}{ccc} (P, \Theta) & \xrightarrow{f} & M \\ \downarrow & & \\ X & & \end{array}$$

$$\Omega_G(M) \xrightarrow{f^*} \Omega_G(P) \xrightarrow{\Theta^*} \Omega(P)_{\text{basic}} \cong \Omega(X)$$

$$\alpha \wedge \beta \wedge \gamma \mapsto f^*(\alpha \wedge \beta \wedge \gamma) \mapsto f^*\alpha(F\Theta) \wedge f^*\beta(\Theta) \wedge f^*\gamma$$

$$X \xrightarrow{(P, \Theta, f)} \mathcal{E}_{\nabla}G \times_G M \quad \Omega(\mathcal{E}_{\nabla}G \times_G M) \xrightarrow{(P, \Theta, f)^*} \Omega(X)$$

Equivariant Differential Cohomology

Equivariant Differential Cohomology à la Redden

Definition

$$\widehat{H}_G^n(M) := \widehat{H}^n(\mathcal{E}_{\nabla}G \times_G M) := \text{hoShv}_{\infty}(\mathcal{E}_{\nabla}G \times_G M, \widehat{H}^n)$$

In other words, $\widehat{\lambda} \in \widehat{H}_G^n(M)$ is a construction

•

$$X \leftarrow (P, \Theta) \xrightarrow{f} M \rightsquigarrow \widehat{\lambda}(P, \Theta, f) \in \widehat{H}^n(X)$$

• If

$$\begin{array}{ccc} (P_1, \Theta_1) & \xrightarrow{\tilde{\varphi}} & (P_2, \Theta_2) \xrightarrow{f} M \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

then $\varphi^* \widehat{\lambda}(P_2, \Theta_2, f) = \widehat{\lambda}(P_1, \Theta_1, f \circ \tilde{\varphi})$.

Equivariant Differential Cohomology

Hexagon diagram à la Redden

The diagram is a hexagon with vertices and arrows as follows:

- Top-left vertex: 0
- Top-right vertex: 0
- Middle-left vertex: $H_G^{n-1}(M; \mathbb{R})$
- Middle-right vertex: $H_G^n(M; \mathbb{R})$
- Bottom-left vertex: $\frac{\Omega_G^{n-1}(M)}{\Omega_G^{n-1}(M)_{\mathbb{Z}}}$
- Bottom-right vertex: 0
- Center vertex: $\hat{H}_G^n(M)$

Arrows and maps:

- $0 \rightarrow H_G^{n-1}(M; \mathbb{R}/\mathbb{Z})$ (top-left to top-middle)
- $H_G^{n-1}(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{-B_G} H_G^n(M; \mathbb{Z})$ (top-middle to top-right)
- $H_G^{n-1}(M; \mathbb{R}) \rightarrow H_G^{n-1}(M; \mathbb{R}/\mathbb{Z})$ (middle-left to top-middle)
- $H_G^{n-1}(M; \mathbb{R}) \rightarrow \frac{\Omega_G^{n-1}(M)}{\Omega_G^{n-1}(M)_{\mathbb{Z}}}$ (middle-left to bottom-left)
- $\frac{\Omega_G^{n-1}(M)}{\Omega_G^{n-1}(M)_{\mathbb{Z}}} \xrightarrow{d_G} \Omega_G^n(M)_{\mathbb{Z}}$ (bottom-left to bottom-middle)
- $\Omega_G^n(M)_{\mathbb{Z}} \rightarrow H_G^n(M; \mathbb{R})$ (bottom-middle to middle-right)
- $H_G^n(M; \mathbb{R}) \rightarrow 0$ (middle-right to bottom-right)
- $H_G^n(M; \mathbb{R}) \rightarrow 0$ (top-right to middle-right)
- $H_G^{n-1}(M; \mathbb{R}/\mathbb{Z}) \rightarrow \hat{H}_G^n(M)$ (top-middle to center)
- $\hat{H}_G^n(M) \rightarrow H_G^n(M; \mathbb{Z})$ (center to top-right)
- $\hat{H}_G^n(M) \rightarrow \Omega_G^n(M)_{\mathbb{Z}}$ (center to bottom-middle)
- $\hat{H}_G^n(M) \rightarrow H_G^n(M; \mathbb{R})$ (center to middle-right)

Commutativity symbols (circles with arrows) are present at the intersections of the following pairs of arrows:

- Between $H_G^{n-1}(M; \mathbb{R}) \rightarrow H_G^{n-1}(M; \mathbb{R}/\mathbb{Z})$ and $H_G^{n-1}(M; \mathbb{R}) \rightarrow \hat{H}_G^n(M)$
- Between $H_G^{n-1}(M; \mathbb{R}/\mathbb{Z}) \rightarrow \hat{H}_G^n(M)$ and $H_G^{n-1}(M; \mathbb{R}/\mathbb{Z}) \rightarrow H_G^n(M; \mathbb{Z})$
- Between $\hat{H}_G^n(M) \rightarrow H_G^n(M; \mathbb{Z})$ and $\hat{H}_G^n(M) \rightarrow H_G^n(M; \mathbb{R})$
- Between $\hat{H}_G^n(M) \rightarrow \Omega_G^n(M)_{\mathbb{Z}}$ and $\hat{H}_G^n(M) \rightarrow H_G^n(M; \mathbb{R})$
- Between $\frac{\Omega_G^{n-1}(M)}{\Omega_G^{n-1}(M)_{\mathbb{Z}}} \rightarrow \hat{H}_G^n(M)$ and $\frac{\Omega_G^{n-1}(M)}{\Omega_G^{n-1}(M)_{\mathbb{Z}}} \rightarrow \Omega_G^n(M)_{\mathbb{Z}}$

Ordinary Differential Cohomology

Cochain Model

$$\begin{array}{ccc} \widehat{C}^n(M) & \longrightarrow & \Omega^n(M; \mathbb{R}) \\ \downarrow & & \downarrow \\ C^n(M; \mathbb{Z}) & \longrightarrow & C^n(M; \mathbb{R}) \end{array}$$

- $\widehat{C}^n(M) = C^n(M; \mathbb{Z}) \times C^{n-1}(M; \mathbb{R}) \times \Omega^n(M; \mathbb{R})$
- $\widehat{d}(c, h, \omega) = (\delta c, c - \int \omega - \delta h, d\omega)$
- $\widehat{H}^\bullet(M) := \text{Ker}(\widehat{d}) / \text{Im}(\widehat{d})$.

Equivariant Differential Cohomology

Cochain Model

$$\begin{array}{ccc} \widehat{C}_G^n(M) & \longrightarrow & \Omega_G^n(M; \mathbb{R}) \\ \downarrow & & \downarrow \\ C_G^n(M; \mathbb{Z}) & \longrightarrow & C_G^n(M; \mathbb{R}) \end{array}$$

- $\widehat{C}_G^n(M) = C_G^n(M; \mathbb{Z}) \times C_G^{n-1}(M; \mathbb{R}) \times \Omega_G^n(M; \mathbb{R})$
- $\widehat{d}_G(c, h, \omega) = (\delta c, c - \int \omega - \delta h, d_G \omega)$
- $\widehat{H}_G^\bullet(M) :=$ cohomology of $(\widehat{C}_G^\bullet(M), \widehat{d}_G)$.

Theorem (Redden, 2016)

The two definitions of $\widehat{H}_G^\bullet(M)$ are naturally isomorphic:

$$\mathrm{hoShv}_\infty(\mathcal{E}_{\nabla} G \times_G M, \widehat{H}^\bullet) \cong \widehat{H}_G^\bullet(M).$$

Main Theorems

Theorem (P. and Redden)

Let $M \in G\text{-Man}$, with G a compact Lie group.

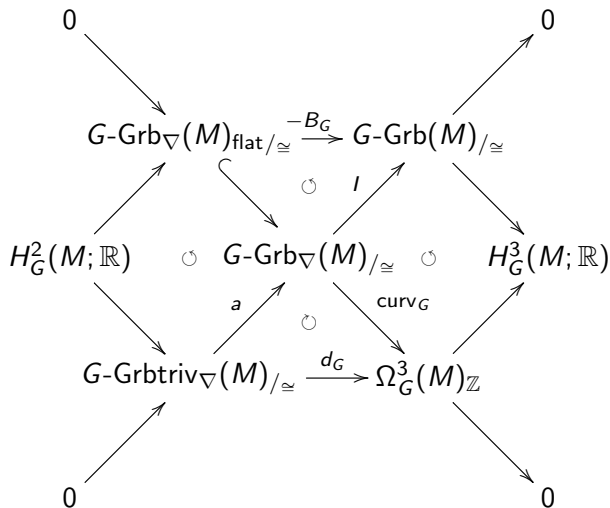
- (1) There is a natural isomorphism of abelian groups

$$G\text{-Grb}_{\nabla}(M)_{/\cong} \xrightarrow{\cong} \widehat{H}_G^3(M).$$

- (2) Furthermore, Redden's equivariant differential cohomology hexagon diagram is naturally isomorphic to ...

Main Theorems

Redden's hexagon diagram in degree 3 with geometric cocycles



Main Theorems

Hexagon diagram à la Redden in degree 3

$$\begin{array}{ccccc} 0 & & & & 0 \\ & \searrow & & & \nearrow \\ & H_G^2(M; \mathbb{R}/\mathbb{Z}) & \xrightarrow{-B_G} & H_G^3(M; \mathbb{Z}) & \\ & \nearrow & \circlearrowleft & \nearrow & \\ H_G^2(M; \mathbb{R}) & & \hat{H}_G^3(M) & & H_G^3(M; \mathbb{R}) \\ & \searrow & \circlearrowleft & \searrow & \\ & \frac{\Omega_G^2(M)}{\Omega_G^2(M)_{\mathbb{Z}}} & \xrightarrow{d_G} & \Omega_G^3(M)_{\mathbb{Z}} & \\ & \nearrow & \circlearrowleft & \nearrow & \\ 0 & & & & 0 \end{array}$$

Main Theorems

Theorem (P. and Redden)

There exists a natural functor

$$G\text{-Grb}_{\nabla}(M) \longrightarrow \text{Grb}_{\nabla}(\mathcal{E}_{\nabla}G \times_G M)$$

which is an equivalence of 2-groupoids.

Thank you!