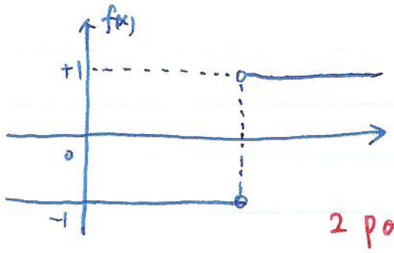
Solution to the Midterm I

#1. (1)



2 points

domain : $\mathbb{R} \setminus \{2012\}$ 3 points

(2) The limit does not exist. Note that $\lim_{x\to 2012} f(x)$ exists if and only if $\lim_{x\to 2012^+} f(x) = \lim_{x\to 2012^-} f(x)$, and in this case

$$\lim_{x\to 2012^+} f(x) = +1 \quad \text{whereas} \quad \lim_{x\to 2012^-} f(x) = -1.$$

#2. For given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{2}$. Then it follows that

$$0 < |x-1| < \delta = \frac{\varepsilon}{2} \Rightarrow 0 < 2|x-1| = |2x-2| < \varepsilon.$$

$$\overset{\text{definition}}{\Longleftrightarrow} \lim_{x\to 1} 2x = 2.$$

7 points

In the given definition, the first inequality in $0 < |x-a| < \delta$ is strict. This means, in this definition, the fact whether the limit exists is irrelevant to the fact whether function is defined at the point where the limit is evaluated. Thus the limit still exists and is 2. 3 points

#3 (See also Quiz 1 solution)

$$\lim_{x\to 2} \frac{x^5-32}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x^4+2x^3+4x^2+8x+16)}{\cancel{x-2}} = 5\times 2^4 = 80.$$

#4

$$\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4} \frac{(\sqrt{x+5}-3)(\sqrt{x+5}+3)}{(x-4)(\sqrt{x+5}+3)} = \lim_{x\to 4} \frac{x+5-9}{(x-4)(\sqrt{x+5}+3)} = \frac{1}{6}.$$

#5. $\lim_{\theta\to0}\frac{2013\sin\theta}{2012\theta} = \lim_{\theta\to0}\frac{2013}{2012}\,\frac{\sin\theta}{\theta} = \frac{2013}{2012}\underbrace{\lim_{\theta\to0}\frac{\sin\theta}{\theta}}_{=1} = \frac{2013}{2012}$

#6. $\lim_{x\to0}\frac{1-e^{-x}}{e^{x}-1} = (-1)\times\lim_{x\to0}\frac{e^{-x}-1}{e^{x}-1} = (-1)\times\lim_{x\to0}\frac{\frac{e^{-x}-1}{-x}\times(-1)}{\frac{e^{x}-1}{x}}$

As $x\to0$, $-x\to0$

$$\downarrow$$

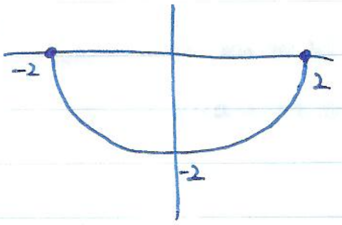
$$= \cancel{(-1)}\times\frac{\lim_{-x\to0}\frac{e^{-x}-1}{-x}\times\cancel{(-1)}}{\lim_{x\to0}\frac{e^{x}-1}{x}} = 1 \quad \text{since } \lim_{x\to0}\frac{e^{x}-1}{x} = 1$$

This is because $t \overset{\text{let}}{=} e^{x}-1$, $e^{x} = 1+t \xrightarrow{\text{take log}} x = \ln(1+t)$,
$\hookrightarrow$ As $x\to0$, $t\to0$.

$$\lim_{x\to0}\frac{e^{x}-1}{x} = \lim_{t\to0}\frac{t}{\ln(1+t)} = \lim_{t\to0}\frac{1}{\frac{1}{t}\ln(1+t)} = \frac{1}{\lim_{t\to0}\ln(1+t)^{1/t}} = \frac{1}{\ln\lim_{t\to0}(1+t)^{1/t}}$$

$$= \frac{1}{\ln e} = 1.$$

#7 (1)



(2) ① On $(-2,2)$, for each $x_0\in(-2,2)$,
$\lim_{x\to x_0} -\sqrt{4-x^2} = -\sqrt{4-x_0^2}$, and hence by definition of continuity, f is continuous at $x=x_0$.

② At $x_0 = \pm2$:

Since $\lim_{x\to+2^-} -\sqrt{4-x^2} = 0$, by the definition of continuity on a closed interval, f is continuous at $x=+2$.

Since $\lim_{x\to-2^+} -\sqrt{4-x^2} = 0$, by the definition of continuity on a closed interval, f is continuous at $x=-2$.

#8. Since $\frac{1}{x}$ is continuous on $\mathbb{R}\setminus\{0\}$, $\sin\frac{1}{x}$ is continuous on $\mathbb{R}\setminus\{0\}$ (clearly $\sin x$: continuous on $\mathbb{R}$)
Hence $x\sin\frac{1}{x}$ is continuous on $\mathbb{R}\setminus\{0\}$, since clearly x: continuous on $\mathbb{R}$ and hence on $\mathbb{R}\setminus\{0\}$. . (Cont'd)

At $x=0$, we claim that f is continuous.

Notice that $-1 \leq \sin\frac{1}{x} \leq 1$ for all $x \in \mathbb{R}\setminus\{0\}$.

$\Rightarrow$ $-x \leq x\sin\frac{1}{x} \leq x$ for all $x \in \mathbb{R}\setminus\{0\}$

Sandwich lemma

$\Rightarrow$ $\lim\limits_{x\to 0} -x = 0 = \lim\limits_{x\to 0} x$ implies

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0 = f(0).$$

continuity follows from this.

#9. Observe that $f(1) = 1$ and $f(-1) = -1$.

Since f is defined on a closed interval and is continuous, by the intermediate value theorem, there exists $c \in [-1,1]$ such that $f(c) = 0 \in [f(-1), f(+1)] = [-1, 1]$.

Accordingly f has a root c in $[-1,1]$.

#10. At $x = \pm 2$ the denominator is nonzero. Hence f has an infinite limit as $x \to \pm 2$, and accordingly the graph of f gets closer to a vertical line $x = \pm 2$, which is, by definition, a vertical asymptote.

#11. $$\lim_{\Delta x\to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x\to 0} \frac{\cancel{x^3} + 3x^2\cancel{\Delta x} + 3x\Delta x^{\cancel{2}} + \Delta x^{\cancel{3}\,2} - \cancel{x^3}}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x\to 0} (3x^2 + 3x\Delta x + \Delta x^2) = 3x^2.$$

#12. $$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \lim_{\theta\to 0} \frac{(1-\cos\theta)\times(1+\cos\theta)}{\theta^2 \times (1+\cos\theta)} \overset{1-\cos^2\theta = \sin\theta}{=} \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta^2(1+\cos\theta)}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta}\frac{\sin\theta}{\theta}\frac{1}{(1+\cos\theta)} \underset{\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1}{=} 1 \times 1 \times \lim_{\theta\to 0}\frac{1}{1+\cos\theta} = 1\times 1\times\frac{1}{2} = \frac{1}{2}.$$

#13. It is clear that $f(x)$ is continuous on $\{x \in \mathbb{R} : x < 2\}$ and on $\{x \in \mathbb{R} : x > 2\}$ (regardless of what $a \in \mathbb{R}$ is).

To have continuity of f at $x = 2$, we need the following is satisfied:

$$\lim_{x \to 2} f(x) = f(2)$$

$\Leftrightarrow$ limit of f exists at $x = 2$

and

the limit is equal to $f(2)$

Note that the limit may not exist depending on the value of a.
To have limit, we need

$$4a = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 8$$

and

$$4a = 8 \Rightarrow a = 2.$$

Now if $a = 2$ we see that $\lim_{x \to 2} f(x)$ exists and is 8.
Since $f(2) = x^3 \big|_{x=2} = 2^3 = 8$

we thus have

$$\lim_{x \to 2} f(x) = 8 = f(2),$$ and verified that

f is continuous at $x = 2$.