Solutions to Midterm II

#1. $\dfrac{d}{d\theta}\tan\theta = \dfrac{d}{d\theta}\left(\dfrac{\sin\theta}{\cos\theta}\right) = \dfrac{(\sin\theta)'\cos\theta - \sin\theta(\cos\theta)'}{\cos^2\theta} = \dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} = \sec^2\theta.$

#2. $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \lim\limits_{\Delta x\to 0}\dfrac{\dfrac{1}{x+\Delta x} - \dfrac{1}{x}}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\dfrac{x-(x+\Delta x)}{(x+\Delta x)\,x}}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{-\cancel{\Delta x}}{(x+\Delta x)x\,\cancel{\Delta x}} = -\dfrac{1}{x^2}$

#3. f has a horizontal tangent line at $x \iff f'(x) = 0$.

Compute: $f'(x) = x^2 - 2$: vanishes at $\boxed{x = \pm\sqrt{2}}$: answer.

#4. $f(\omega) = 6\omega - 5\ln\omega$. At $\omega = 1$, $f(1) = 6$.

$f'(\omega) = 6 - \dfrac{5}{\omega}$. $f'(1) = 1$.

$\left\langle \begin{array}{l} \text{slope: } f'(1) = 1 \\ \text{point: } (1,6) \end{array}\right. \Rightarrow y = 1\cdot(x-1) + 6 = x+5.$: answer.

#5. $y = \frac{1}{2}e^z - 3\sin z$. At $z = \pi$, $y\big|_{z=\pi} = \frac{1}{2}e^{\pi}$.

$y' = \frac{1}{2}e^z - 3\cos z$ $\qquad y'\big|_{z=\pi} = \frac{1}{2}e^{\pi} + 3.$

$\left\langle \begin{array}{l} \text{slope: } \frac{1}{2}e^{\pi}+3 \\ \text{point: } (\pi, \frac{1}{2}e^{\pi}) \end{array}\right. \Rightarrow y = \left(\tfrac{1}{2}e^{\pi}+3\right)(x-\pi) + \tfrac{1}{2}e^{\pi}$

$= \left(\tfrac{1}{2}e^{\pi}+3\right)x - \tfrac{\pi}{2}e^{\pi} - 3\pi + \tfrac{1}{2}e^{\pi}$

$\underline{y = \left(\tfrac{1}{2}e^{\pi}+3\right)x + \left(\tfrac{1-\pi}{2}\right)e^{\pi} - 3\pi.}$ answer.

#6. Since a, b are constants, we know ax^3 and x^2+b are differentiable everywhere on $\mathbb{R}$. To make $f(x) = \begin{cases} ax^3 & (x \le 2) \\ x^2+b & (x > 2) \end{cases}$ differentiable everywhere on $\mathbb{R}$,

f must be differentiable (and hence continuous as well) at $x = 2$.

① differentiability at $x=2$: the limit of $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ should exist as $\Delta x \to 0$ either from the left or from the right.

left limit $= \lim\limits_{\Delta x\to 0^-}\dfrac{f(2+\Delta x) - f(2)}{\Delta x} = \lim\limits_{\Delta x\to 0^-}\dfrac{a(2+\Delta x)^3 - a\cdot 2^3}{\Delta x} = \lim\limits_{\Delta x\to 0^-}\dfrac{\cancel{a\cdot 2^3} + a\cdot 3\cdot 2^2\cancel{\Delta x} + a\cdot 3\cdot 2\cdot\Delta x^{\cancel{2}} + a\Delta x^{\cancel{3}\,2} - \cancel{a\cdot 2^3}}{\cancel{\Delta x}}$

$= 12a.$

right limit $= \lim\limits_{\Delta x\to 0^+}\dfrac{f(2+\Delta x) - f(2)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{(2+\Delta x)^2 + b - 8a}{\Delta x} \underset{\uparrow}{=} \lim\limits_{\Delta x\to 0}\dfrac{(\cancel{4} + 4\cancel{\Delta x} + \Delta x^{\cancel{2}\,1}) + (\cancel{8a} - \cancel{4}) - \cancel{8a}}{\cancel{\Delta x}}$

(*) See below

$= 4.$

Hence $12a = 4 \iff a = \frac{1}{3}.$

② Continuity at $x=2$: the left and the right limit must agree at least.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} a x^3 = 8a = 4+b \quad \Rightarrow \quad \boxed{b = 8a-4} \ ⊛$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2+b = 4+b$$

From above $a = \frac{1}{3}$ $\Rightarrow$ $b = 8\cdot\frac{1}{3} - 4 = -\frac{4}{3}$ ✓

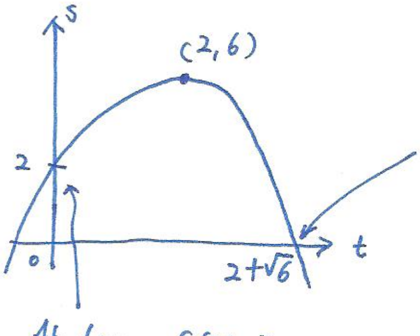
#7. (1) $f'(\alpha) = \dfrac{(5\alpha-2)'(\alpha^2+1)-(5\alpha-2)(\alpha^2+1)'}{(\alpha^2+1)^2} = \dfrac{5(\alpha^2+1)-(5\alpha-2)\cdot 2\alpha}{(\alpha^2+1)^2}$.

(2) $y' = (3\beta^3+4\beta)'(e^{2\beta}+1)(\ln 7\beta^2+1) + (3\beta^3+4\beta)(e^{2\beta}+1)'(\ln 7\beta^2+1)$
$+ (3\beta^3+4\beta)(e^{2\beta}+1)(\ln 7\beta^2+1)'$

$= (9\beta^2+4)(e^{2\beta}+1)(\ln 7\beta^2+1) + (3\beta^3+4\beta)\, 2\cdot e^{2\beta}\cdot(\ln 7\beta^2+1)$
$+ (3\beta^3+4\beta)(e^{2\beta}+1)\cdot \underbrace{\dfrac{(7\beta^2)'}{7\beta^2}}_{\frac{2}{\beta}}$

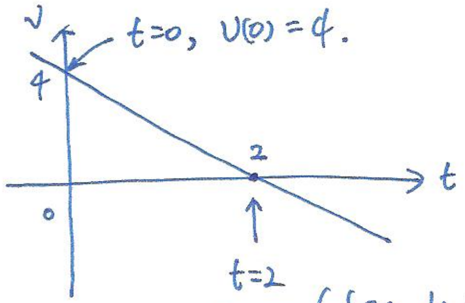
#8. $v(t) = s'(t) = -2t+4$
$a(t) = s''(t) = -2$

#9. $s(t) = -t^2+4t+2 = -t^2+4t-4+4+2 = -(t-2)^2+6$



At $t=0$, $s(0)=2$.

$0 = s(t) = -(t-2)^2+6$
$(t-2)^2 = 6$
$t = 2 \pm \sqrt{6}$ → $\boxed{2+\sqrt{6}}$: makes a good sense
→ $2-\sqrt{6}$ does not make sense ($t<0$).



$t=2$
$v=0$ (See this is when the above parabola reaches at maximum in $s(t)$.)

#10. $3e^{xy} - x = 0 \xrightarrow{\text{implicit diff.}} 3e^{xy}\frac{d}{dx}(xy) - 1 = 0$

$\Leftrightarrow 3e^{xy}\left(y + x\frac{dy}{dx}\right) - 1 = 0$

$3xe^{xy}\frac{dy}{dx} = 1 - 3e^{xy}\cdot y$

$\frac{dy}{dx} = \frac{1-3e^{xy}\cdot y}{3xe^{xy}}$ (makes sense only if denominator $\neq 0$)

Now $\begin{cases} \text{Slope: } \left.\frac{dy}{dx}\right|_{(3,0)} = \frac{1-3e^{3\cdot 0}\cdot 0}{3\cdot 3e^{3\cdot 0}} = \frac{1}{9} \\ \text{Point: } (3,0) \end{cases}$ $\Rightarrow y = \frac{1}{9}(x-3) = \underline{\frac{1}{9}x - \frac{1}{3}}$

answer.

#11. For $y = \arcsin x$, we wish to find $\frac{dy}{dx}$.

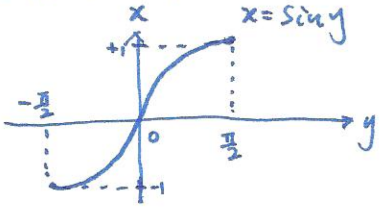
Inverse function theorem says $\frac{dy}{dx} = \frac{1}{dx/dy}$ if $\frac{dx}{dy} \neq 0$.

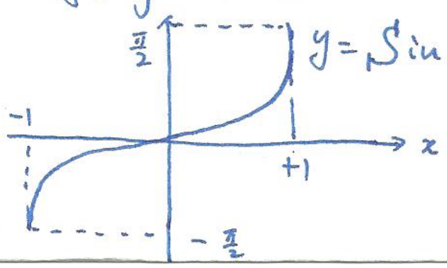
So we find $\frac{dx}{dy}$ first. By the definition of inverse function,

$y = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin y.$

So $\frac{dx}{dy} = \cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}.$

Note that, from the following graph of $y = \sin^{-1} x$,





We observe that on the domain $(-1,1)$ of $y = \arcsin x$, $1-x^2$ is strictly positive. Hence we may drop $-\sqrt{1-x^2}$. Also on $(-1,1)$,

$\frac{dx}{dy} = \sqrt{1-x^2} \neq 0$ for any $x \in (-1,1)$.

Thus, by the inverse function theorem,

$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\sqrt{1-x^2}}.$ ✓

#12. (1) $f'(x) = -\cos(\cos x)\sin x$ (chain rule)

(2) $y = \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln(x+2)$

$$y' = \frac{1}{3(x-2)} - \frac{1}{3(x+2)}.$$

#13. $V = \frac{1}{3}\pi r^2 h$, $\frac{dr}{dt} = 2\ in/min$ $h = 3r$.

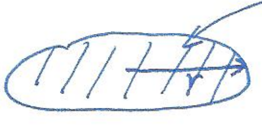
$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^2 \cdot 3r\right) = \frac{d}{dt}\left(\pi r^3\right) = 3\pi r^2 \frac{dr}{dt}$$

$$= 3\pi\,(6\ \text{inch})^2 \cdot 2\ \text{inch}/\text{min} = 3\pi\ 36\,[\text{inch}]^2 \cdot 2\,[\text{inch}]/[\text{min}]$$

$$= \underline{\underline{216\pi}}\ \text{inch}^3/\text{min}.$$

#14.

$A = \pi r^2$



Given $\frac{dr}{dt} = 3\ m/s$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi \cdot 6m \cdot 3\ m/s = \underline{\underline{36\pi}}\ m^2/s$$

#15. $\lim_{x\to a} f(x) = f(a)$: def. of continuity

<u>Claim</u> $\lim_{x\to a}[f(x) - f(a)] = 0$

$$\lim_{x\to a} f(x)-f(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot x-a = f'(a)\underbrace{\lim_{x\to a} x-a}_{=0} = 0. \checkmark$$

because f: differentiable at $x=a$.

#16. No. $f(x) = |x|$ defined on $\mathbb{R}$.

At $x=0$ f is continuous (obvious) but not differentiable:

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{-x-0}{x-0} = -1 \quad \text{whereas}$$

$\neq$ ← i.e. $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist.

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x-0}{x-0} = +1.$$

Hence f: not diff. at $x=0$. ✓