Addenda — Lesson 2

MAT175 Section B402

August 30th, 2012

Perhaps not all of you are very comfortable with the following application of additive formula of sine or cosine functions. Let's have a quick review about it. Suppose you are given the following function:

$$a \sin x + b \cos x$$

The following is the technique which all of you should master. First, pull out $\sqrt{a^2+b^2}$ to obtain:

$$\sqrt{a^2+b^2}\Big(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\Big). \tag{1}$$

If you imagine a triangle that has the length of its hypotenuse $\sqrt{a^2+b^2}$. Then there will be two angles α and β in the right-angled triangle, satisfying $\alpha+\beta=\pi/2$. If the following is the case:

$$\sin\alpha = \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \cos\alpha = \frac{b}{\sqrt{a^2+b^2}}, \tag{2}$$

we simultaneously have:

$$\sin\beta = \frac{b}{\sqrt{a^2+b^2}} \quad \text{and} \quad \cos\beta = \frac{a}{\sqrt{a^2+b^2}}. \tag{3}$$

Now if we use (2), (1) becomes

$$\sqrt{a^2+b^2}(\sin\alpha\sin x + \cos\alpha\cos x) = \sqrt{a^2+b^2}\cos(x-\alpha), \tag{4}$$

and if we use (3), (1) becomes

$$\sqrt{a^2+b^2}(\cos\beta\sin x + \sin\beta\cos x) = \sqrt{a^2+b^2}\sin(x+\beta). \tag{5}$$

Now look at the first summand of $a\sin x + b\cos x$. If you look up the graph of this, a is the amplitude of the oscillation of the sine curve, and $x \in \mathbb{R}$ is the angle or the horizontal coordinate. Similarly for $b\cos x$. Now, since you are adding these two terms, at some $x \in \mathbb{R}$, $a\sin x$ and $b\cos x$ are of the same sign, and at some other $x \in \mathbb{R}$, they will have different signs. Now in the former case, the oscillation will become larger, and in the latter case, the oscillation will become smaller. But since both summands are *periodic* with the *same* period($=2\pi$), you can expect that $a\sin x + b\cos x$ will also be periodic with the period 2π. A question here: What's the definition of the period of a function?(Look up, if you are not sure) So it is natural to expect that you can represent $a\sin x + b\cos x$ as a single sine or cosine function, albeit the amplitude or the phase may be changed.(Question: What is the *phase* of a trigonometric function? What's the phase of $A\sin(x-\delta)$)? Now we note that the following formula: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ which you should have learned from the precalculus gives an explicit way to combine two summands $a\sin x + b\cos x$ into a single sine or cosine function. It doesn't matter you combine it into sine or cosine. As you see in (4) and (5), only phases look different, but these functions are actually not different. Why? Recall $\sin(x+(\pi/2)-\alpha) = \cos(x-\alpha)$.