Addenda — Lesson 4

MAT175 Section B402

September 6th, 2012

We now understand the $\epsilon - \delta$ definition of limit:

$$\lim_{x\to a} f(x) = L \Leftrightarrow \text{For every } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } 0 < |x-a| < \delta \text{ implies } |f(x) - L| < \epsilon. \tag{1}$$

Now we prove that if a limit exists, it must be unique.

Proposition 1. *Let $f : I \to \mathbb{R}$ is a function defined on an interval. Let $a \in I$. For any real numbers L_1 and L_2, if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$ hold, then $L_1 = L_2$.*

Proof. We want to prove that $L_1 = L_2$, and this equivalent to the following: $|L_1 - L_2| < \epsilon$ for any $\epsilon > 0$. Now by our assumption, $L_1 = \lim_{x\to a} f(x) = L_2$. That is, for every $\epsilon_1 > 0$ there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ implies $|f(x) - L_1| < \epsilon_1$, and simultaneously, for every $\epsilon_2 > 0$ there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ implies $|f(x) - L_2| < \epsilon_2$. Hence if we take $\delta = \min\{\delta_1, \delta_2\}$, it follows that $|L_1 - L_2| \le |L_1 - f(x)| + |f(x) - L_2| < \epsilon_1 + \epsilon_2$ whenever $0 < |x - a| < \delta$. Note that we used the triangle inequality $|a + b| \le |a| + |b|$. Since ϵ_1 and ϵ_2 were arbitrary, we may as well take $\epsilon_1 = \epsilon_2 = \epsilon/2$ where ϵ is arbitrary. Hence we established that $|L_1 - L_2| < \epsilon$ for an arbitrary $\epsilon > 0$. □

Exercise. Using the definition (1), prove that any $M \in [-1, 1]$ is not equal to $\lim_{x\to 0} \sin \frac{1}{x}$.

Exercise. Let $f : I \to \mathbb{R}$ is a function defined on an interval. Let $a \in I$. Assume $f(a) > 0$ and $\lim_{x\to a} f(x) = f(a)$. Explain why we can find an open interval $(c, d) \subset I$ containing a such that $f(x)$ is positive on (c, d).

Exercise. First study thoroughly Example 6 and Example 7 in p.73. (1) Explain that, in cases of each function given in these examples, the choice of δ is depending only on ϵ and not on x.
(2) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to a} f(x) = f(a)$ for all $a \in \mathbb{R}$, which has δ depending both on ϵ and a, in the argument for proving $\lim_{x\to a} f(x) = f(a)$ by using $\epsilon - \delta$ arguments. Explain why your example is satisfying all these requirements.

Comments on Homework 1

Section 1.6 Exercises 59, 61 and 63 asks, by using the fact that each of $\ln x$ and e^x is the inverse of the other, to simplify the following: 59. $\ln e^{x^2}$ 61. $e^{\ln(5x+2)}$ 63. $e^{\ln\sqrt{x}}$. Recall that if $g : Y \to X$ is an inverse function of a function $f : X \to Y$, $g \circ f = \mathbf{1}_X$ and $f \circ g = \mathbf{1}_Y$. Now let $f(x) = e^x$ and $g(x) = \ln x$. Note that $g \circ f$ is defined on $\mathbb{R}$, but $f \circ g$ is defined only on $(0, \infty)$. Hence $g \circ f(x) = \ln e^x = x$ for all $x \in \mathbb{R}$, and $f \circ g(x) = e^{\ln x} = x$ for all $x \in (0, \infty)$. Now it follows that 59. $\ln e^{x^2} = x^2$ for all $x \in \mathbb{R}$; 61. $e^{\ln(5x+2)} = (5x + 2)$ for all $x > -2/5$; 63. $e^{\ln\sqrt{x}} = \sqrt{x}$ for all $x > 0$.