Quiz 1 Solution

#1. For any given $\varepsilon > 0$, we take $0 < \delta \le \frac{\varepsilon}{2}$. It follows that

$$0 < |x-1| < \delta \implies |2x+3-5| = 2|x-1| < 2\cdot\frac{\varepsilon}{2} = \varepsilon. \quad (15\text{ points})$$

Even if $\{-1\}$ is subtracted from the domain $\mathbb{R}$, the limit still exists, since, in the given definition, the first inequality of $0 < |x-a| < \delta$ is strict, and accordingly the definition is independent from the fact whether the domain of a given function contains the point where the limit is evaluated. (5 points)

#2. (1) First we factorize:

$$\begin{aligned} x^5 - 32 &= (x-2)(x^4 + ax^3 + bx^2 + cx + d) \\ &= x^5 + ax^4 + bx^3 + cx^2 + dx \\ &\quad - 2x^4 - 2ax^3 - 2bx^2 - 2cx - 2d \\ &\overset{(*)}{=} x^5 + (a-2)x^4 + (b-2a)x^3 + (c-2b)x^2 + (d-2c)x - 2d \end{aligned}$$

Since the equality should hold for all $x \in \mathbb{R}$, the above equality is an identity. (deduct 5 points if this was not mentioned.)

We compare coefficients:

$a-2=0$ | $a=2$
$b-2a=0$ | $b=4$
$c-2b=0$ | $c=8$
$d-2c=0$ | $d=16$
$-2d=-32$

Hence $\displaystyle\lim_{x\to2}\frac{x^5-32}{x-2} = \lim_{x\to2}\frac{\cancel{(x-2)}(x^4+2x^3+4x^2+8x+16)}{\cancel{x-2}}$

$\displaystyle = \lim_{x\to2} x^4+2x^3+4x^2+8x+16 = 5\cdot2^4 = \underline{\underline{80}}.$

(2) $$\lim_{\delta x\to0}\frac{\sin(x+\delta x)-\sin x}{\delta x} = \lim_{\delta x\to0}\frac{\sin x\cos\delta x + \cos x\sin\delta x - \sin x}{\delta x}$$

$$= \sin x\left(\lim_{\delta x\to0}\frac{\cos\delta x - 1}{\delta x}\right) + \cos x\lim_{\delta x\to0}\frac{\sin\delta x}{\delta x}$$

$= \underline{\cos x}$, since $\displaystyle\underline{\lim_{\delta x\to0}\frac{\sin\delta x}{\delta x} = 1}$ and $\displaystyle\underline{\lim_{\delta x\to0}\frac{\cos\delta x-1}{\delta x} = 0}$

Don't need to prove these, but if these ~~were~~ were not mentioned, deduct 10 points.

#3 $$\lim_{x\to4}\frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to4}\frac{(\sqrt{x+5}-3)(\sqrt{x+5}+3)}{(x-4)(\sqrt{x+5}+3)} = \lim_{x\to4}\frac{\cancel{x+5-9}}{\cancel{(x-4)}(\sqrt{x+5}+3)} = \underline{\underline{\frac{1}{6}}}.$$

#4. $$\lim_{\theta\to0}\frac{\cos\theta\tan\theta}{\theta} = \lim_{\theta\to0}\frac{\cos\theta\frac{\sin\theta}{\cos\theta}}{\theta} = \lim_{\theta\to0}\frac{\sin\theta}{\theta} = \underline{\underline{1}}.$$