Homework 1: Discussion on Section 1.2. Exercise 77

Fall 2013 MAT175 Section C401[19514]

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Let $A = (b, c)$, $B = (-a, 0)$ and $C = (a, 0)$ as given in the problem(compare with the problem given in p.17 of 5th edition).

The idea is as follows. Sides $\overline{AB}$ and $\overline{CA}$ of the triangle are fixed. Imagine that you draw straightlines perpendicular to $\overline{AB}$ and $\overline{CA}$. They must meet at one point. Now how many pairs of such lines perpendicular to $\overline{AB}$ and $\overline{CA}$ can you draw? There are infinitely many of them. In fact, those pairs are parametrized by the intersection point. Now what if you have an extra straightline that is perpendicular to $\overline{BC}$ and meeting with the other lines perpendicular to $\overline{AB}$ and $\overline{CA}$ at their intersection point? Then among infinitely many pairs of lines perpendicular to $\overline{AB}$ and $\overline{CA}$, there is only one pair that can make this happen. Hence if we let $P = (x, y)$ be the intersection those three vertical lines, then the fact that P is on the vertical line passing through $A = (b, c)$ forces that lines perpendicular to $\overline{AB}$ and $\overline{CA}$ must meet at a point lying on the segment joining A and $(b, 0)$, and those conditions for those lines perpendicular to $\overline{AB}$ and $\overline{CA}$ should allow us to find what is the y-coordinate of $P = (x, y)$.

Here is the details: Since $P = (x, y)$ is lying on the vertical line passing through $A = (b, c)$, we immediately get $x = b$. Now note that

- The line passing through $P = (b, y)$ and $B = (-a, 0)$ is perpendicular to the line having slope $\frac{-c}{a-b}$ — which is the slope of $\overline{CA}$.
- The line passing through $P = (b, y)$ and $C = (a, 0)$ is perpendicular to the line having slope $\frac{c}{b+a}$ — which is the slope of $\overline{AB}$.

We have

$$\frac{y}{b+a}\frac{-c}{a-b} = -1$$
$$\frac{-y}{a-b}\frac{c}{b+a} = -1$$

which are in fact the same equations.[1] Hence we get

$$y = \frac{(a+b)(a-b)}{c}.$$

Therefore

$$P = (x, y) = (b, \frac{(a+b)(a-b)}{c}).$$

[1]You may regard this situation in a way that we got a redundant information, but one should note that one can prove that P is already determined using lines perpendicular to $\overline{AB}$ and $\overline{CA}$ and *passing through* C *and* B, respectively. i.e. being two vertical lines in a triangle should be sufficient in determining P.