Sample Final Exam Solution

#1. $y = \sqrt{3} + \frac{3}{x^2} + \sqrt{x}$

$y' = -6x^{-3} + \frac{1}{2}x^{-\frac{1}{2}}$

$= -\frac{6}{x^3} + \frac{1}{2\sqrt{x}}.$

#2. $r(t) = \frac{\pi}{t} + \frac{t}{\pi} - \pi t$

$r'(t) = -\frac{\pi}{t^2} + \frac{1}{\pi} - \pi.$

#3. point: $(0, y|_{x=0} = 0)$
Slope: $y'|_{x=0} = 1 + \cos x|_{x=0} = 2.$

$\Longrightarrow$ tangent line: $y = 2x.$

#4. point: $(-4, 1)$
slope: $\frac{dy}{dx}\Big|_{(-4,1)} = \frac{4}{3}.$

$\frac{x^2}{2} + y^3 = 9$

Implicit diff. ↓

$\frac{2x\,dx}{2} + 3y\,dy = 0.$

$\frac{dy}{dx} = -\frac{x}{3y} \underset{(-4,1)}{=} -\frac{(-4)}{3} = \frac{4}{3}.$

$\Longrightarrow$ tangent line: $y - 1 = \frac{4}{3}(x+4)$

$y = \frac{4}{3}x + \frac{16}{3} + \frac{3}{3}$

$\boxed{y = \frac{4}{3}x + \frac{19}{3}}$

#5. $\frac{dz}{dx} = (x^2)'e^x + x^2(e^x)'$

$= 2xe^x + x^2e^x$

$= (2+x)xe^x$

#6. $p(\theta) = \ln(2 + \sin\theta)$

$p'(\theta) = \frac{(2+\sin\theta)'}{2+\sin\theta} = \frac{\cos\theta}{2+\sin\theta}$

#7. $\lim_{t\to\infty} \frac{\sin t}{1-e^t} = \textcircled{0}$

because

$\frac{-1}{1-e^t} \le \frac{\sin t}{1-e^t} \le \frac{1}{1-e^t}$

$\xrightarrow[\text{lemma } t\to\infty]{\text{Sandwich}}$ $0 \le \lim_{t\to\infty} \frac{\sin t}{1-e^t} \le 0.$

#8. $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \textcircled{6}$

#9. $\lim_{x\to\infty} \frac{x^2+10}{9x^3+1} = 10.$

#10. $\frac{dA}{dt} = 3$ $\quad A = s^2$

when $A = 49 = s^2$,
$s = 7.$

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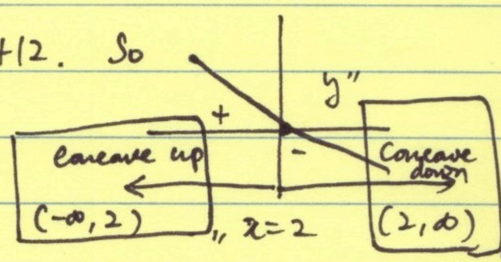
$3 = \frac{dA}{dt} = 2s\frac{ds}{dt} = 2\cdot 7\cdot\frac{ds}{dt}$

$\frac{ds}{dt} = \frac{3}{14}$ inch/sec.

#11. $y'' = -6x + 12.$ So



Inflection point: $(2, y|_{x=2}) = (2, 28)$

#12. $x(t) = t^3 + 3t$.

$v(t) = x'(t) = 3t^2 + 3 \overset{t=5}{\Longrightarrow} \boxed{78}$.

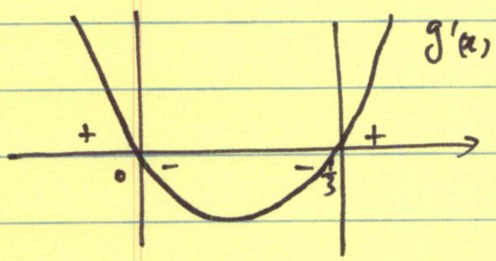
$a(t) = x''(t) = 6t \overset{t=5}{\Longrightarrow} \boxed{30}$.

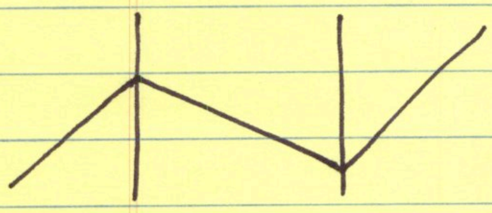
#13. $g(x) = 2x^3 - x^2$

$g'(x) = 6x^2 - 2x = 0$

$\Leftrightarrow x(6x-2) = 0$

$x = 0$ or $\frac{1}{3}$.





Candidates: $x = 0, \frac{1}{3}, 1$

$g(0) = 0$

$g(\frac{1}{3}) = 2 \cdot \frac{1}{27} - \frac{1}{9} = -\frac{1}{27}$

$g(1) = 1$

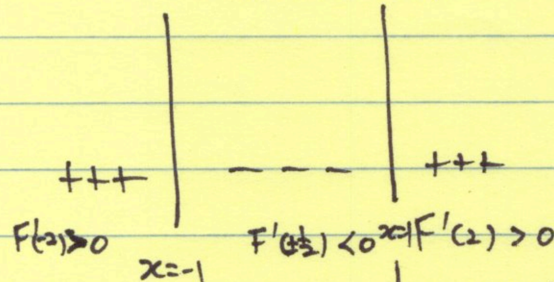
when $x=1$ $g(x)$ has maximum $\boxed{1}$

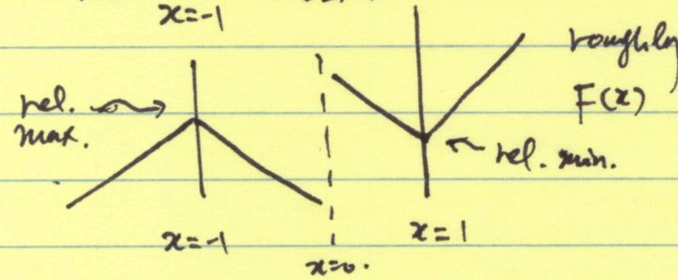
when $x = \frac{1}{3}$, $g(x)$ has minimum $\boxed{-\frac{1}{27}}$

#14. $F(x) = x + \frac{1}{x}$

$F'(x) = 1 - \frac{1}{x^2} = 0$

$\Leftrightarrow x = \pm 1$





F(x) has a relative maximum -2 at $x=-1$ and a relative minimum 2 at $x=1$.

#15. $f(x) = 3x - x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h) - (x+h)^2 - (3x - x^2)}{h}$$

$$= \lim_{h\to 0} \frac{3x + 3h - x^2 - 2xh - h^2 - 3x + x^2}{h}$$

$$= \lim_{h\to 0} \frac{3h - 2xh - h^2}{h} = 3 - 2x$$

#16. For $Q(x)$ to be continuous on $\mathbb{R}$, it must be continuous at $x=0$, and for this, $\lim_{x\to 0^+} Q(x) = \lim_{x\to 0^-} Q(x)$

$\left(= \lim_{x\to 0^+} \sqrt{x}\right)$ $\left(= \lim_{x\to 0^-} k - e^x\right)$

So, $0 = k - 1 \Leftrightarrow \boxed{k=1}$

We check that $\lim_{x\to 0} Q(x) = Q(0)$

$\lim_{x\to 0} Q(x) = 0$, $Q(0) = 1 - e^0 = 0$

Hence $Q(x)$ is continuous at $x=0$.

Clearly $Q(x)$ is continuous elsewhere.