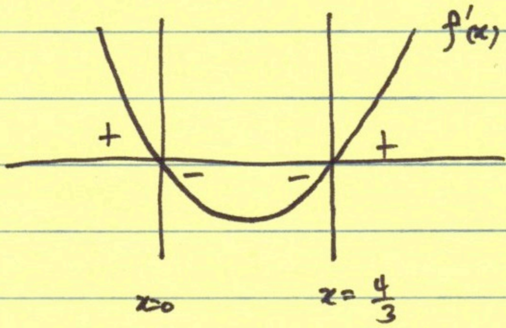
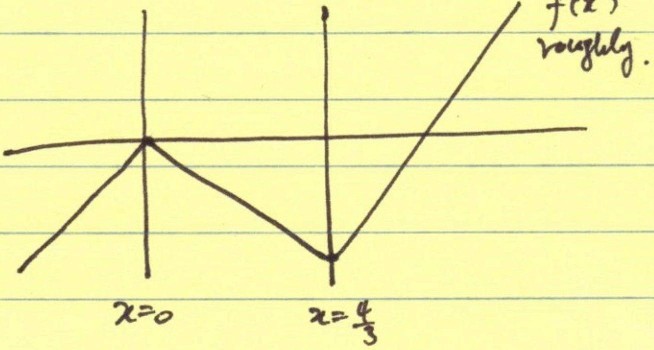
Solutions to Midterm Exam 3

#1 $f(x) = 2x^3 - 4x^2$

$f'(x) = 6x^2 - 8x$

$f'(x) = 0 \iff 6x^2 - 8x = 0 \iff x = 0$ or $x = \frac{4}{3}$





Candidates $x=0 \quad x=\frac{4}{3} \quad x=2 \quad x=-1$

$f(0)=0, \ f(\frac{4}{3})= \quad f(2)=0, \ f(-1)=-6$

$$f(\tfrac{4}{3}) = 2\cdot\frac{4^3}{3^3} - 4\cdot\frac{4^2}{3^2} = \frac{2\cdot 4^3 - 12\cdot 4^2}{3^3} = \frac{(2-3)\cdot 4^3}{3^3} < 0.$$

Since $\left|-\frac{64}{27}\right| < |-6|$,

$$\boxed{\begin{array}{l}\text{minimum is } -6 \text{ at } x=-1 \\ \text{maximum is } 0 \text{ at } x=0 \text{ or } 2\end{array}}$$

#9. $$\lim_{x\to\infty} \frac{x^3+2x+1}{x^2+x+1} = \lim_{x\to\infty} \frac{1+\frac{2}{x^2}+\frac{1}{x^3}}{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}} = \infty$$

#10. $$\lim_{x\to\infty} \frac{2x^2+x+2}{3x^2-x+1} = \lim_{x\to\infty} \frac{2+\frac{1}{x}+\frac{2}{x^2}}{3-\frac{1}{x}+\frac{1}{x^2}} = \frac{2}{3}$$

#11.

$$-1 \le \cos t \le 1$$

$$\Rightarrow -\frac{1}{1-e^t} \le \frac{\cos t}{1-e^t} \le \frac{1}{1-e^t}$$

By Squeeze theorem,

since $\lim_{t\to\infty} \pm\frac{1}{1-e^t} = 0$,

we have $\lim_{t\to\infty} \frac{\cos t}{1-e^t} = 0$

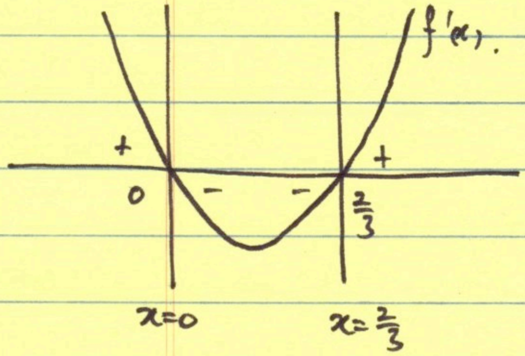
#12. $$\lim_{x\to\infty} \frac{\sqrt{x}+2}{x+1} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}+\frac{2}{x}}{1+\frac{1}{x}} = 0$$

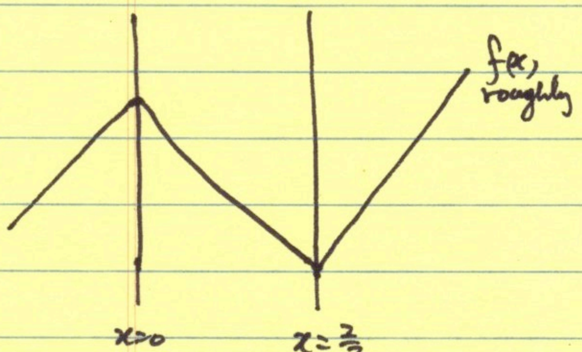
#2 $f(x) = x^3 - x^2$.

$f'(x) = 3x^2 - 2x$.

$f'(x) = 0 \iff x(3x-2) = 0$

$\iff x = 0$ or $x = \frac{2}{3}$.





Candidates: $x=0$ $x=\frac{2}{3}$ $x=1$

$f(0)=0$ $f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^2$

$= \frac{8-12}{27} < 0$

$f(1) = 0$.

f has absolute maximum 0 at $x = 0$ or 1

and absolute minimum $-\frac{4}{27}$ at $x = \frac{2}{3}$

on the interval $[0,1]$.

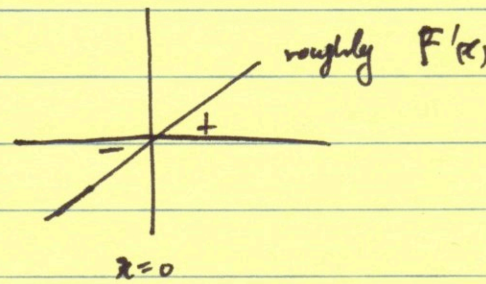
#3 $F(x) = x^4 + 5x^2 + 6$.

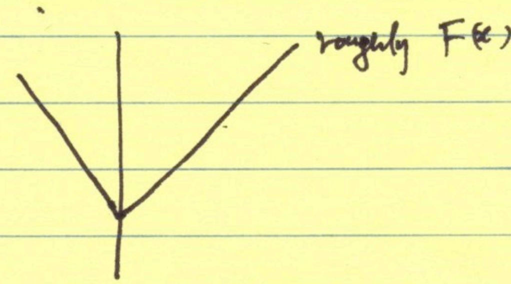
$F'(x) = 4x^3 + 10x$.

$F'(x) = 0 \iff 4x^3 + 10x = 0$

$\iff x(\underbrace{4x^2+10}_{>0}) = 0$

$x = 0$.



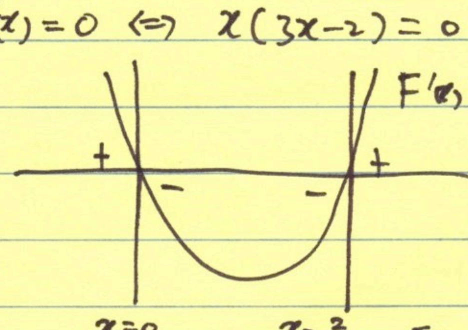


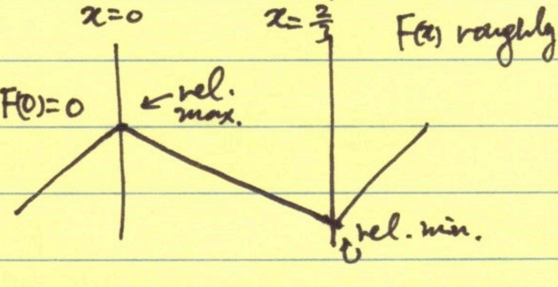
$F(x)$ has a relative minimum at $x=0$, and the rel. min. is $F(0) = 6$.

#4 $F(x) = x^3 - x^2$

$F'(x) = 3x^2 - 2x$

$F'(x) = 0 \iff x(3x-2) = 0 \iff x = 0$ or $\frac{2}{3}$.





$F(x)$ has a relative maximum $F(0)=0$ at $x=0$, and a relative minimum $F\left(\frac{2}{3}\right) = -\frac{4}{27}$ at $x = \frac{2}{3}$.

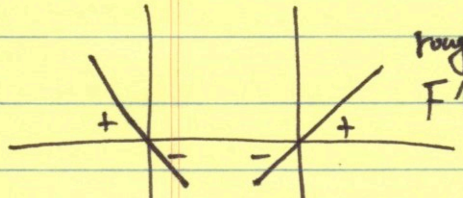
#5

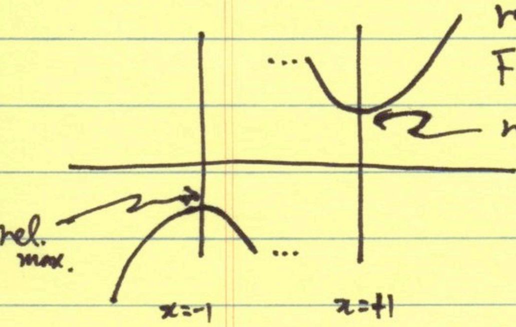
$F(x) = 2x + \frac{2}{x}$

$F'(x) = 2 - \frac{2}{x^2}$

$F'(x) = 0 \Leftrightarrow 2 - \frac{2}{x^2} = 0$

$\Leftrightarrow x = \pm 1.$



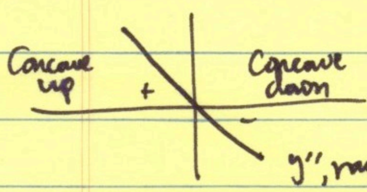


F(x) has a relative maximum -4 when $x = -1$, and a relative minimum 4 when $x = +1$.

#6 $y = -x^3 + x^2 + 2x - 1$

$y' = -3x^2 + 2x + 2$

$y'' = -6x + 2 \overset{\text{put}}{=} 0 \Leftrightarrow x = \frac{1}{3}.$



y is concave up on all $x < \frac{1}{3}$
concave down on all $x > \frac{1}{3}$

when $x = \frac{1}{3}$, inflection point is

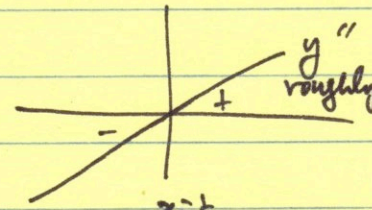
$\left(\frac{1}{3}, y|_{x=\frac{1}{3}}\right) = \left(\frac{1}{3}, -\frac{1}{27} + \frac{1}{9} + \frac{2}{3} - 1\right)$

$= \boxed{\left(\frac{1}{3}, -\frac{7}{27}\right)}.$

#7 $y = x^3 - x^2$

$y' = 3x^2 - 2x$

$y'' = 6x - 2. \overset{\text{put}}{=} 0 \quad x = \frac{1}{3}.$



inflection point

$\left(\frac{1}{3}, y|_{x=\frac{1}{3}}\right) = \boxed{\left(\frac{1}{3}, \frac{-2}{27}\right)}$

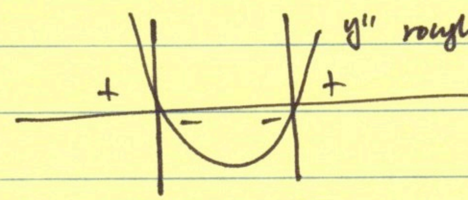
y is concave up on all $x > \frac{1}{3}$
Concave down on all $x < \frac{1}{3}$

#8 $y = x^4 - 4x^3.$

$y' = 4x^3 - 12x^2$

$y'' = 12x^2 - 24x \overset{\text{put}}{=} 0$

$x = 0$ or $x = 2$



y is concave up on all $x < 0$ or $x > 2$

Concave down on $0 < x < 2$

Inflection points

$(0, y|_{x=0}) = \boxed{(0, 0)}$

$(2, y|_{x=2}) = \boxed{(2, -16)}$