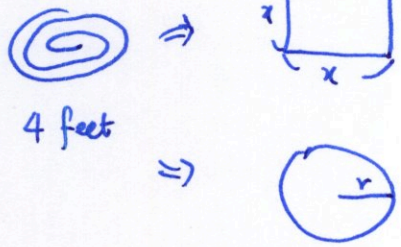
Example Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

Sol:



4 feet

	Perimeter	Area
⇒ square	$4x$	x^2
⇒ circle	$2\pi r$	πr^2

perimeter condition: $4 = 4x + 2\pi r \Rightarrow x = \frac{4 - 2\pi r}{4} = 1 - \frac{1}{2}\pi r$.

Area $A(r) = x^2 + \pi r^2 = \left(1 - \frac{1}{2}\pi r\right)^2 + \pi r^2$

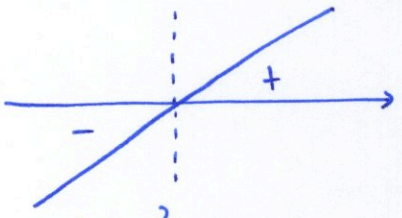
$= 1 - \pi r + \frac{1}{4}\pi^2 r^2 + \pi r^2$

$= 1 - \pi r + \left(\frac{1}{4}\pi^2 + \pi\right) r^2 \quad \cdots \circledast$

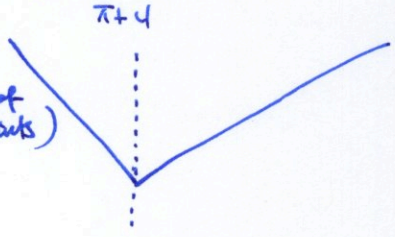
$A'(r) = -\pi + \left(\frac{1}{2}\pi^2 + 2\pi\right) r \overset{\text{put}}{=} 0$

$r = \frac{1}{\frac{1}{2}\pi + 2} = \frac{2}{\pi + 4}$.

graph of $-\pi + \left(\frac{1}{2}\pi^2 + 2\pi\right) r$



So $A(r)$ may have maximum either at $r = 0$ or $2\pi r = 4$ (i.e. at one of endpoints) ($\Leftrightarrow r = \frac{2}{\pi}$).



From $\circledast$ above,

when $r = 0$, $A(0) = 1$

$r = \frac{2}{\pi}$, $A\left(\frac{2}{\pi}\right) = 1 - \pi \cdot \frac{2}{\pi} + \left(\frac{1}{4}\pi^2 + \pi\right)\frac{4}{\pi^2} = 1 - \pi\frac{2}{\pi} + \frac{1}{4}\pi^2\frac{4}{\pi^2} + \pi\frac{4}{\pi^2}$

$= 1 - 2 + 1 + \frac{4}{\pi} = \frac{4}{\pi}$.

Hence the maximum area is obtained as $\frac{4}{\pi}$ when $x = 0$ and $r = \frac{2}{\pi}$ (i.e. when the entire wire is used to bound a circle). ✓