Minor Calculation Mistake : -1

Exam III
MATH 155 Section 08
December 17th, 2015. 6:20PM–8:20PM

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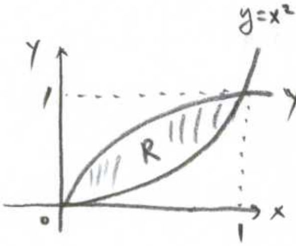
Instructions: Please clearly write your name above. This exam is closed-book and closed-note. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Write all details explicitly. Answers without justifications and/or calculation steps may receive no score. Hand-in this exam sheets and other sheets which contain your work to be graded. Cross out everything which you do not want them to be graded.

Total 100 points. 10 points each unless specified otherwise.

1. (5 points) Let R be the region in the xy-plane bounded by the curves $y = \sqrt{x}$ and $y = x^2$. Set up an integral which equals the volume of the solid formed by rotating the region R around the x-axis. <u>Do not</u> evaluate the integral.

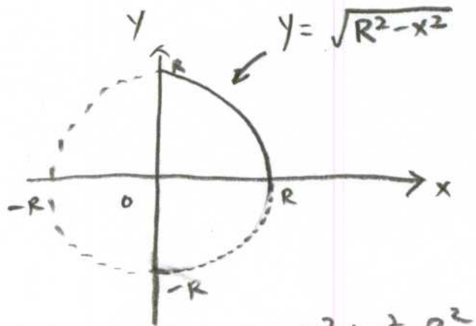


$$V_x = \int_0^1 \pi\left[(\sqrt{x})^2 - (x^2)^2\right]dx$$

$$= \int_0^1 \pi(x - x^4)\,dx.$$

No partial credit.

2. (10 points) Prove by using integration that the surface area of the sphere with radius R is $4\pi R^2$.



Surface Area obtained by 1/4 of the circle with radius R about x-axis

$\hookrightarrow A = \int_0^R 2\pi y \sqrt{1 + y'^2}\, dx$

+4 Correct Setup

$x^2 + y^2 = R^2$

$2x\,dx + 2y\,dy = 0$

$\frac{dy}{dx} = -\frac{x}{y}$

+2 correct procedure ①

So $A = \int_0^R 2\pi y \sqrt{1 + \frac{x^2}{y^2}}\, dx = \int_0^R 2\pi \sqrt{y^2 + x^2}\, dx = 2\pi R \int_0^R dx = 2\pi R^2$.

+4 Correct procedure ②

This is the area of a hemisphere. Thus the surface area of the given sphere is $4\pi R^2$.

3. (10 points) Calculate the following integral:

$$I = \int e^{3x} \cos 2x dx.$$

Each Correct integration by parts +3

Correct answer +4.

$$I = e^{3x} \cdot \frac{1}{2}\sin 2x - \int 3e^{3x} \cdot \frac{1}{2} \sin 2x \, dx$$

$$= \frac{1}{2} e^{3x} \sin 2x - \frac{3}{2}\left[e^{3x}\left(-\frac{1}{2}\right)\cos 2x - \int 3e^{3x}\left(-\frac{1}{2}\right)\cos 2x \, dx \right]$$

$$= \frac{1}{2} e^{3x} \sin 2x + \frac{3}{4} e^{3x} \cos 2x - \frac{9}{4} \int e^{3x} \cos 2x \, dx$$

$$\left(1+\frac{9}{4}\right) I = \frac{1}{4}\left(2 e^{3x} \sin 2x + 3 e^{3x} \cos 2x\right)$$

$$\therefore \; I = \frac{1}{13}\left(2 e^{3x} \sin 2x + 3 e^{3x} \cos 2x\right) + C$$

4. (5 points each) Evaluate or show divergence:

(1)

$$I = \int_1^\infty \frac{1}{\sqrt{x}}dx$$

$$I = \underbrace{\lim_{R\to\infty} \int_1^R}\frac{1}{\sqrt{x}}dx = \lim_{R\to\infty} \frac{1}{1-\frac{1}{2}} x^{-\frac{1}{2}+1}\Big|_1^R = \lim_{R\to\infty} 2\sqrt{x}\Big|_1^R = \lim_{R\to\infty}(2\sqrt{R}-2)$$

$$= \infty - 2 = \infty.$$

missing this expression -1

Correct integral +3

Correct answer +2.

(2)

$$I = \int_0^\infty e^{-x}dx$$

$$I = \lim_{R\to\infty} -e^{-x}\Big|_0^R = \lim_{R\to\infty} -e^{-R}+1 = 0+1=1.$$

Same rubric as above.

5. (5 points each) Compute the limit of the sequence or show divergence:

(1)

$$\lim_{k\to\infty} \frac{e^k}{k^2}.$$

L'Hôpital's rule

$$\lim_{k\to\infty} \frac{e^k}{k^2} = \lim_{k\to\infty} \frac{e^k}{2k} = \lim_{k\to\infty} \frac{1}{2} e^k = \infty.$$

Correct idea +3

Correct answer +2

(2)

$$\lim_{n\to\infty} \frac{\cos n}{n}.$$

Squeeze theorem: Since $-1 \le \cos n \le 1$,

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n},$$

and from $\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$, by Squeeze theorem, $\lim_{n\to\infty} \frac{\cos n}{n} = 0$.

Same rubric as above

(3)

$$S = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{3}{2^k}.$$

Geometric Series

first term ($k=0$) $a=3$

ratio $r=\frac{1}{2}$.

$$S = \frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = 6$$

Same rubric as above.

6. Given an infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1},$$

show that the series is convergent using indicated methods:

(1) (3 points) The comparison test. (You can use $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent without proof.)

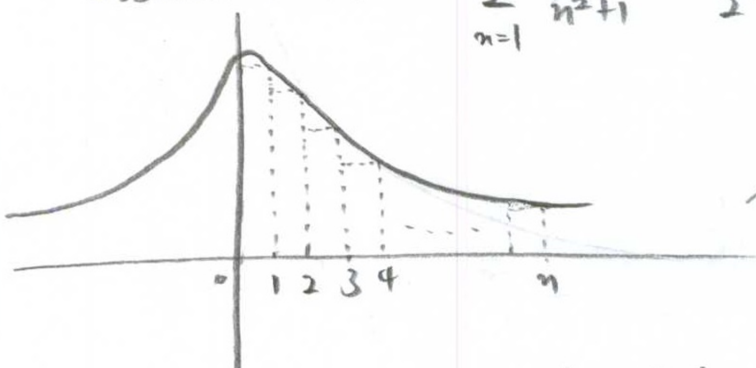
Observe that $\frac{1}{n^2+1} < \frac{1}{n^2}$ for all $n \geq 1$.

By the Comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$ ✓

Correct use of the Comparison test : +3

(2) (7 points) The integral test. (You should compute a finite integral you need for comparison.)

Observe that: $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{1}{5} + \cdots < \int_0^{\infty} \frac{1}{1+x^2}\, dx$



$$= \tan^{-1} x \Big|_0^{\infty} = \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}.$$

Hence by the integral test, it Converges.

Correct application of integral test +4

Correct integral +3

7. (5 points) Show that the alternating Harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent.

Alternating Series Test with $a_n = \frac{1}{n}$

(1) $a_n \geq a_{n+1}$ for all n

$$\frac{1}{n} \geq \frac{1}{n+1} \quad ✓$$

(2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$ ✓

Hence by the Alternating Series Test, it converges.

8. (10 points) Write down the degree 4 Taylor polynomial centered at 0:

$$p_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k$$

for given $f(x) = 1 + \cos x$.

each correct term +2

$f'(x) = -\sin x \overset{\text{at } x=0}{=\!=} 0$

$f''(x) = -\cos x \overset{\text{at } x=0}{=\!=} -1$

$f'''(x) = \sin x \overset{\text{at } x=0}{=\!=} 0$

$f''''(x) = \cos x \overset{\text{at } x=0}{=\!=} 1$

$$P_4(x) = 2 + 0\cdot x^1 + \frac{(-1)}{2!}x^2 + 0\cdot x^3 + \frac{1}{4!}x^4$$

$$= 2 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

9. (10 points) Find the interval of convergence of the power series:

$$\sum_{n=2}^{\infty} \frac{5(x-2)^n}{n-1}.$$

(Clearly mention whether your final answer is a(n) open, half-open, or closed interval!)

By ratio test, the series is (abs.) convergent if

$$\lim_{n\to\infty} \left| \frac{5(x-2)^{n+1}}{(n+1)-1} \Big/ \frac{5(x-2)^n}{n-1} \right| = \lim_{n\to\infty} |x-2| \left|\frac{n-1}{n}\right| = |x-2| < 1$$

i.e. $< x < 3$. 5pts up to here

① when $x=1$

3pts

$$\sum_{n=2}^{\infty} \frac{5\cdot(-1)^n}{n-1} = 5\sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$$

: alternating series with $a_n = \frac{1}{n-1}$. ① $a_n = \frac{1}{n-1} \geq \frac{1}{n} = a_{n+1}$ ② $\lim_{n\to\infty} a_n = 0$

② when $x=3$

2pts.

$$\sum_{n=2}^{\infty} \frac{5}{n-1} > 5\sum_{n=2}^{\infty} \frac{1}{n} = \infty$$

Hence the interval of convergence is $1 \leq x < 3$.

10. (1) (5 points) Let C be a circle of radius 2 centered at $(2,0)$. Write the equation of C in the polar coodinate.

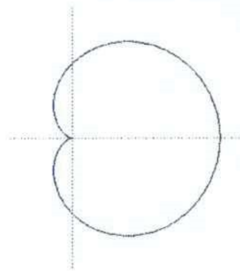
$C: (x-2)^2+y^2 = 2^2$

$\Leftrightarrow x^2-4x+4+y^2=4$

$\Leftrightarrow x^2+y^2 = 4x$

$\Leftrightarrow r^2 = 4r\cos\theta$

$\Leftrightarrow r = 4\cos\theta$.



(2) (10 points) Calculate the enclosed area by the cardioid $r = 1 + \cos\theta$ depicted as above.

$$A = \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} 1 + 2\cos\theta + \underbrace{\cos^2\theta}_{=\frac{1+\cos 2\theta}{2}} d\theta$$

Correct Setup +5

$$= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]\Big|_0^{2\pi}$$

$$= \frac{1}{2}(2\pi + 0 + \pi + 0) - 0 = \frac{3}{2}\pi.$$

Correct answer +5.