MAT 155 Project VII Spring 2014

**Derivatives**

**Do not work on this file.** This is just the list of problems. Open your own file and save it regularly as yourname7.mws and as yourname7b.mws. Sign your name as a comment at the top of your file by backspacing in front of the prompt and typing. Also write Project VII and the names of any other students you are working with. Don't forget to number your problems and to type restart at the beginning of each problem.

**If you cannot complete a problem, go on to the next one and return to the problem later.** You can **get a new prompt** by selecting the prompt button right below the word "spreadsheet". It has the symbol "[>" on it. You must hit enter on every line of the problem in order, including the restart line, to review what you've done for the Maple program.

**If you cannot recall a command** from a previous lab you may consult the command index which can be opened up as a second window. The name of the file with the command index is 155.00.00.html.

**Problem 1:** Recall how to define a function:

```
> f:= x -> x^2;
```

$$x \to x^2 \quad (1)$$

Recall that the derivative of a function is the limit as h approaches 0 of the "difference quotient":

```
> (f(x+h)-f(x))/h;
```

$$\frac{(x+h)^2 - x^2}{h} \quad (2)$$

Here we are using an h instead of the usual Δx.
You can find this limit by expanding the numerator:

```
> expand(f(x+h)-f(x));
```

then cancelling the h by hand and using direct substitution. Do this and then check your answer by using the limit command:

$$h^2 + 2\,h\,x \quad (3)$$

```
> limit( ( f(x+h)-f(x) )/h , h=0 );
```

$$2\,x \quad (4)$$

Now it is easy to take the limit as h goes to zero, and you can see that the answer is 2x.
That is f'(x)=2x.

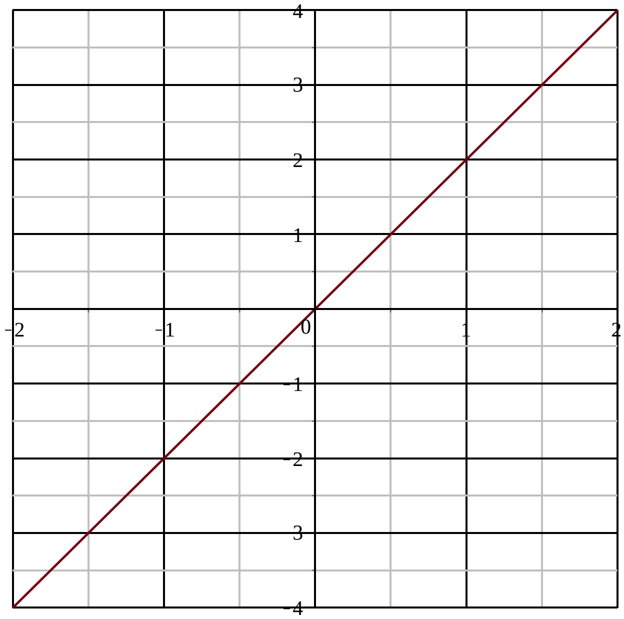
The computer can do all this work to find the derivative in one step if you use the command D:

```
> D(f);
```

$$x \to 2\,x \quad (5)$$

Notice that D(f) is a function. So to graph it we must type D(f)(x):

```
> plot( D(f)(x), x=-2..2);
```




Now define the function g(x)= x^3-x. Compute its derivative using the difference quotient by expanding the numerator, dividing out the h and using direct substitution. Then check using the limit command and finally use the D command. Then graph D(f)(x).

> $g := x \rightarrow x^3 - x;$

$$x \rightarrow x^3 - x \tag{6}$$

> $expand(g(x+h) - g(x));$

$$h^3 + 3\,h^2\,x + 3\,h\,x^2 - h \tag{7}$$

> $Q := h \rightarrow \frac{(h^3 + 3\,h^2\,x + 3\,h\,x^2 - h)}{h};$

$$h \rightarrow \frac{h^3 + 3\,h^2\,x + 3\,h\,x^2 - h}{h} \tag{8}$$

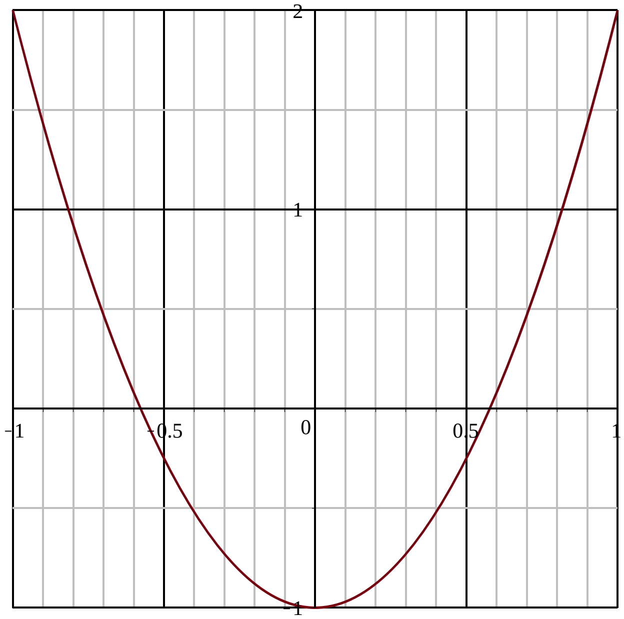
> $limit(\mathrm{Q}(h), \mathrm{h} = 0);$

$$3\,x^2 - 1 \tag{9}$$

> $\mathrm{D}(g);$

$$x \rightarrow 3\,x^2 - 1 \tag{10}$$

> $plot(\mathrm{D}(g)(x), x = -1\,..1);$

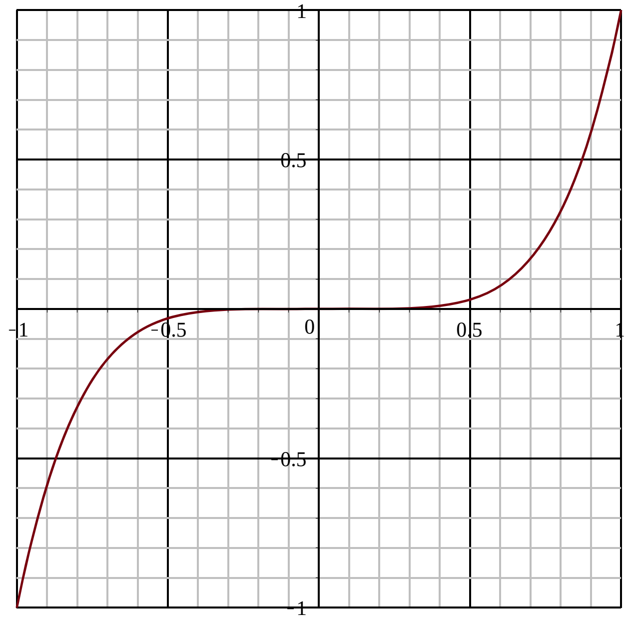



**Problem 2:** Don't forget to restart. Define a function f which maps x to x^5. Graph it.

$> f := x \rightarrow x^5;$

$$x \rightarrow x^5 \tag{11}$$

$> plot(f(x), x = -1 ..1);$




Now we want to find the tangent line at x=2. Instead of using a loop command and graphing a lot of secant lines, we can just use the fact that the difference quotient (f(x+h)-f(x))/h measures the slope of the secant line through the points (x,f(x)) and ((x+h),f(x+h)). So the slope of the tangent line can be found by taking the limit of this difference quotient as h goes to 0 (then x+h goes to x). In problem 1 we saw that this limit can be done quickly using the D(f)(x) command. So to find the slope of the tangent line to the graph of f at (2,f(2)), we find the derivative of f using the D command and find its value at 2:

```
> D(f)(2);
```

This should be the slope of the graph of f at (2, f(2)). So call it m by using the following command:

```
> m:=D(f)(2);
```

Now define the line through (2,4) with the slope m as a function L:

```
> L:= x -> m*(x-2)+f(2);
```

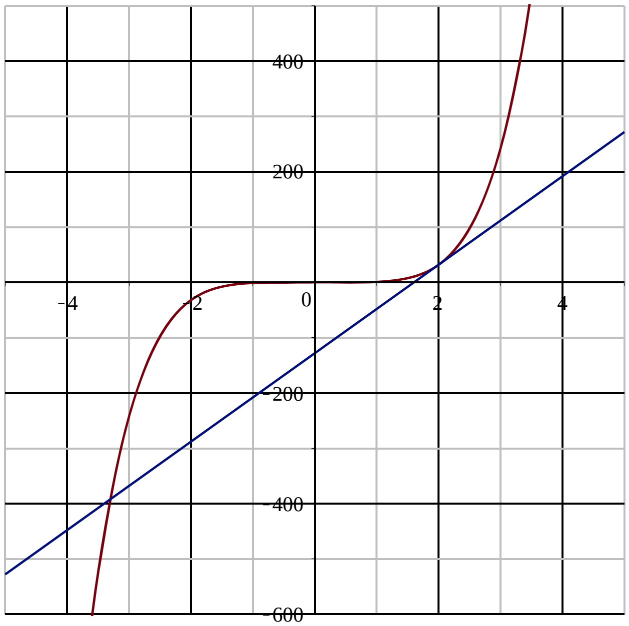
$$80 \tag{12}$$

$$80$$

$$x \rightarrow m\,(x-2) + f(2)$$

Then graph both f and L together using the command:

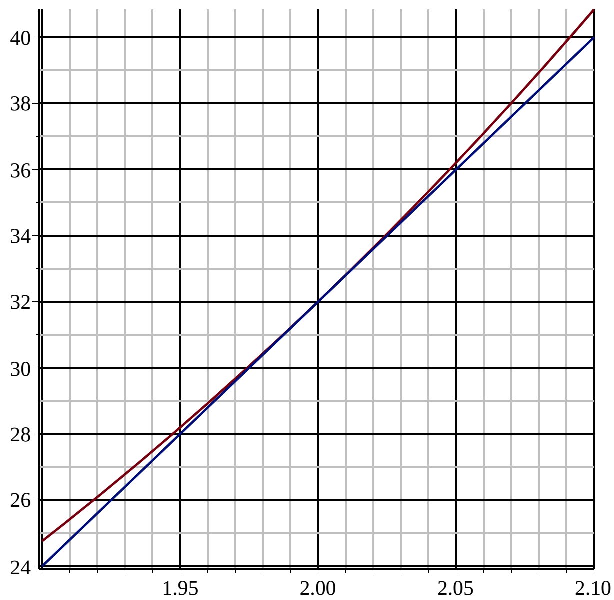
```
> plot({f(x),L(x)}, x=-5..5);
```




You can see that you have found the tangent line from the graph.

Now examine the graphs near where they meet by plotting f(x) and L(x) together on smaller and smaller intervals around x=2. Try x in [1,3], x in [1.9, 2.1] and x in [1.99, 2.01]. Note that the two graphs look almost the same! For this reason L(x) is called the "**Linear Approximation of f(x) near x=2**".

> $plot(\{f(x), L(x)\}, x = 1.9..2.1);$




**Problem 3:** Don't forget to restart. Take the function f which takes x to x^2+2x+1 and find its tangent lines at x=-2, x=2 and x=0. Be sure to use different variables to represent the slope (like firstm, secondm and thirdm) and to give the lines different names (like firstL, secondL and thirdL). *In computers it is common to use entire words for functions and variables. In math we cannot do this because when letters are placed next to each other it is assumed that they are multiplied.*

> $f := x \rightarrow x^2 + 2\,x + 1;$

$$x \rightarrow x^2 + 2\,x + 1 \tag{13}$$

> $firstm := \mathrm{D}(f)(-2);$

$$-2 \tag{14}$$

> $f(-2);$

$$1 \tag{15}$$

> $L1 := x \rightarrow -2 \cdot (x + 2) + 1;$

$$x \rightarrow -2\,x - 3 \tag{16}$$

> $secondm := \mathrm{D}(f)(2);$

$$6 \tag{17}$$

> $thirdm := \mathrm{D}(f)(0);$

$$2 \tag{18}$$

> $f(2); f(0);$

$$9 \tag{19}$$
$$1$$

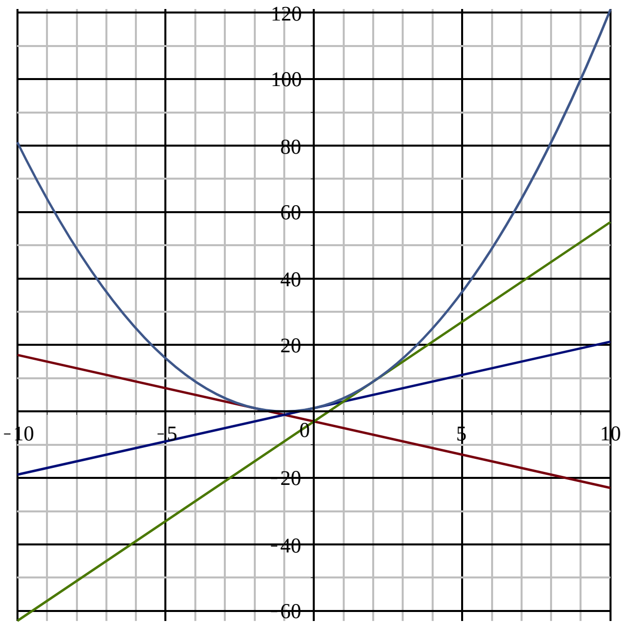
> $L2 := x \to 6 \cdot (x-2) + 9;\ L3 := x \to 2 \cdot (x) + 1;$

$$x \to 6\,x - 3$$
$$x \to 2\,x + 1 \qquad \textbf{(20)}$$

> ?

Now graph all three tangent lines together with f(x) by including all of them in the brackets{}, and write a comment discussing which lines are increasing and decreasing. Recall that a graph is increasing if it goes up when travelling from left to right and it is decreasing when it goes down from left to right.

> $plot(\{f(x), L1(x), L2(x), L3(x)\}, x = -10..10);$

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Exploration:
**A:** A ball is thrown straight up in the air and its height is described by the formula h(t)= -4.9 t^2 +100t +2 meters where t measures the time in seconds. The velocity is then described by the function v(t)=D(h)(t).
What is the initial velocity v(0)? When a ball is thrown straight up in the air, its velocity starts out positive then right before the ball starts to drop back down the velocity is zero and when the ball goes down the velocity is negative. To find out how high the ball goes before it falls, solve for when the velocity is zero (v(t)=0) and then substitute the answer into the height equation h(t). To find out when the ball hits the ground solve for h(t)=0 and only accept a positive answer (time starts at 0).
**B:** Write a loop program which tells you t, h(t),v(t) for t from 1 to 20 (consult the command index if

necessary).
Be sure to save your work before running the loop.
**C:** The last line of the loop's output should be

$$20, 42.0, -96.0 \quad \textbf{(21)}$$

This means that at time t=20 seconds, the ball is 42 meters high and falling downward at 96 meters per second.
Using this information you can find the tangent line, L(t), to h(t) at t=20 seconds. This tangent line is a linear approximation of h and can be used to estimate the values of h near t=20. To see this plot L(t) and f(t) for t in [19,20]. If we fsolve(L(t)=0, t=19..21), we can see when the linear approximation of the height hits zero.
This should be a good estimate for when the height itself hits zero fsolve(h(t), t=19..20); This idea has been used since Newton.

```
> h := t -> -4.9 t^2 + 100 t + 2;
```

$$t \rightarrow (-1) \cdot 4.9\, t^2 + 100\, t + 2 \quad \textbf{(22)}$$

```
> v_init : D(h)(0);
```

$$100. \quad \textbf{(23)}$$

```
> t_top := solve(D(h)(t) = 0, t);
```

$$10.20408163 \quad \textbf{(24)}$$

```
> h(t_top);
```

$$512.2040817 \quad \textbf{(25)}$$

```
> solve(h(t) = 0, t);
```

$$-0.01998043832, 20.42814370 \quad \textbf{(26)}$$

```
> for t from 1 to 20 do t, h(t), D(h)(t); od;
```

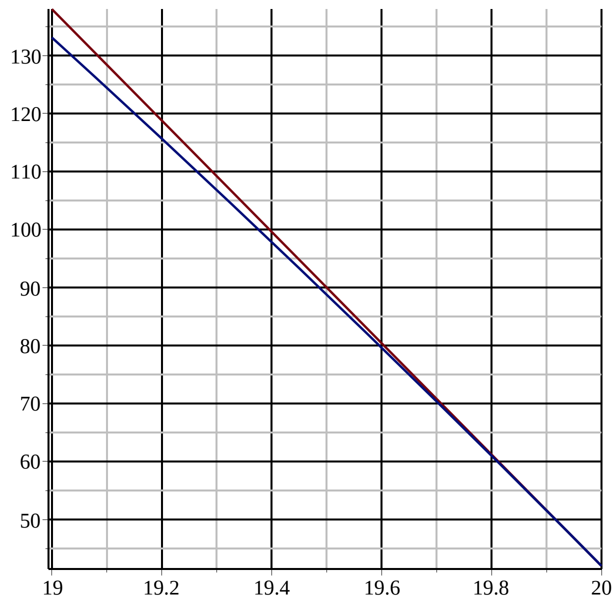
$$1, 97.1, 90.2 \quad \textbf{(27)}$$

2, 182.4, 80.4
3, 257.9, 70.6
4, 323.6, 60.8
5, 379.5, 51.0
6, 425.6, 41.2
7, 461.9, 31.4
8, 488.4, 21.6
9, 505.1, 11.8
10, 512.0, 2.0
11, 509.1, −7.8
12, 496.4, −17.6
13, 473.9, −27.4
14, 441.6, −37.2
15, 399.5, −47.0
16, 347.6, −56.8
17, 285.9, −66.6
18, 214.4, −76.4
19, 133.1, −86.2
20, 42.0, −96.0

```
> L := t -> -96 (t - 20) + 42;
```

$$t \rightarrow -96\, t + 1962 \quad \textbf{(28)}$$

```
> plot( {h(x), L(x)}, x = 19..20);
```




> $fsolve(L(x) = 0, x = 19..21);$

$$20.43750000 \tag{29}$$

> ?