# MATH 156 LAB 10

*Topic 1: Integration by parts.*

## BYUNG DO PARK

```
> with(student):
```

Maple has a command that performs integration by parts. It is intparts(f, u), where $u$ is our choice for integration in the formula $\int u \left( \frac{\partial}{\partial x} v \right) dx$=uv-$\int \left( \frac{\partial}{\partial x} u \right) v\, dx$. Here $f$ is an integral (not a function) previously defined. Maple does not automatically decide which should be $u$ and which should be $v'$. You have to decide which function to choose. Example:

```
> E1:=Int(exp(2*x)*(2*x+3), x);
```

$$E1 := \int e^{2x} (2x+3)\, dx$$

```
> E2:=intparts(E1, 2*x+3);
```

$$E2 := \frac{1}{2} e^{2x} (2x+3) - \left( \int e^{2x}\, dx \right)$$

```
> value(E2);
```

$$\frac{1}{2} e^{2x} (2x+3) - \frac{1}{2} e^{2x}$$

**Perform the integration by parts in the integral $\int x^{10} \ln(x)\, dx$.**

**Verify your answer by asking Maple to integrate directly**.

```
> E1:=Int(x^(10)*ln(x), x);
```

$$E1 := \int x^{10} \ln(x)\, dx$$

```
> E2:=intparts(E1, ln(x));
```

$$E2 := \frac{1}{11} \ln(x)\, x^{11} - \left( \int \frac{1}{11} x^{10}\, dx \right)$$

```
> value(E2);
```

$$\frac{1}{11} \ln(x)\, x^{11} - \frac{1}{121} x^{11}$$

```
> value(E1);
```

$$\frac{1}{11} \ln(x)\, x^{11} - \frac{1}{121} x^{11}$$

Many times we are forced to integrate by parts repeatedly. We do this one step at a time. Example: Calculate the integral $\int e^{2x} \cos(3x)\, dx$.

```
Lint:=Int(exp(2*x)*cos(3*x), x);
```

$$Lint := \int e^{2x} \cos(3x)\, dx$$

```
> Mint:=intparts(Lint, exp(2*x));
```

$$Mint := \frac{1}{3} e^{2x} \sin(3x) - \left(\int \frac{2}{3} e^{2x} \sin(3x)\, dx\right)$$

We define the new integral to be Nint:

```
> Nint:=Int(2/3*exp(2*x)*sin(3*x), x);
```

$$Nint := \int \frac{2}{3} e^{2x} \sin(3x)\, dx$$

```
> Oint:=intparts(Nint, exp(2*x));
```

$$Oint := -\frac{2}{9} e^{2x} \cos(3x) - \left(\int \left(-\frac{4}{9} e^{2x} \cos(3x)\right) dx\right)$$

Here we see that we got back the integral we started with with coefficient 4/9 in front. The result is

```
> Lint:=1/3*exp(2*x)*sin(3*x)-Oint;
```

$$Lint := \frac{1}{3} e^{2x} \sin(3x) + \frac{2}{9} e^{2x} \cos(3x) + \int \left(-\frac{4}{9} e^{2x} \cos(3x)\right) dx$$

```
> Lint:=(1+4/9)^(-1)*(1/3*exp(2*x)*sin(3*x)+2/9*exp(2*x)*cos(3*x));
```

$$Lint := \frac{3}{13} e^{2x} \sin(3x) + \frac{2}{13} e^{2x} \cos(3x)$$

We can check the answer by redefining the integral and integrating directly:

```
> lint:=Int(exp(2*x)*cos(3*x), x);
```

$$lint := \int e^{2x} \cos(3x)\, dx$$

```
> value(lint);
```

$$\frac{3}{13} e^{2x} \sin(3x) + \frac{2}{13} e^{2x} \cos(3x)$$

**Perform two integrations by parts to compute the integral** $\int \left(x^2 + 5x + 9\right) e^{2x}\, dx$**. Check your answer with direct**

integration with Maple.

```
> E1:=Int((x^2+5*x+9)*e^(2*x), x);
```

$$E1 := \int (x^2+5\,x+9)\,e^{2\,x}\,dx$$

```
> E2:=intparts(E1,x^2+5*x+9);
```

$$E2 := \frac{1}{2}\,\frac{(x^2+5\,x+9)\,e^{2\,x}}{\ln(e)} - \left(\int \frac{1}{2}\,\frac{(2\,x+5)\,e^{2\,x}}{\ln(e)}\,dx\right)$$

```
> E21:=(Int((1/2)*(2*x+5)*e^(2*x)/ln(e), x));
```

$$E21 := \int \frac{1}{2}\,\frac{(2\,x+5)\,e^{2\,x}}{\ln(e)}\,dx$$

```
> E22:=intparts(E21,(2*x+5));
```

$$E22 := \frac{1}{4}\,\frac{(2\,x+5)\,e^{2\,x}}{\ln(e)^2} - \left(\int \frac{1}{2}\,\frac{e^{2\,x}}{\ln(e)^2}\,dx\right)$$

```
> value(E22);
```

$$\frac{1}{4}\,\frac{(2\,x+5)\,e^{2\,x}}{\ln(e)^2} - \frac{1}{4}\,\frac{e^{2\,x}}{\ln(e)^3}$$

```
> E2new:=(1/2)*(x^2+5*x+9)*e^(2*x)/ln(e)-((1/4)*(2*x+5)*e^(2*x)/ln
  (e)^2-(1/4)*e^(2*x)/ln(e)^3);
```

$$E2new := \frac{1}{2}\,\frac{(x^2+5\,x+9)\,e^{2\,x}}{\ln(e)} - \frac{1}{4}\,\frac{(2\,x+5)\,e^{2\,x}}{\ln(e)^2} + \frac{1}{4}\,\frac{e^{2\,x}}{\ln(e)^3}$$

```
> value(E1);
```

$$\frac{1}{4}\,\frac{(2\,x^2\ln(e)^2+10\ln(e)^2\,x+18\ln(e)^2-2\ln(e)\,x-5\ln(e)+1)\,e^{2\,x}}{\ln(e)^3}$$

**Evaluate the integral** $\int x \ln\left(x+\sqrt{1+x^2}\right)\,dx$**. If you try to compute the integral directly you will see that Maple gets stuck and does not know what to do. You will need two integrations by parts and the second is tricky. Ask Maple to simplify the second integrand and to use** $u = x$**. This shows that sometimes we need to help Maple in its work. Do not always rely on Maple!**

```
> E1:=int(x*ln(x+sqrt(x^2+1)), x); value(E1);
```

$$E1 := \int x\ln\left(x+\sqrt{x^2+1}\right)\,dx$$

$$\int x\ln\left(x+\sqrt{x^2+1}\right)\,dx$$

```
> E2:=intparts(E1,ln(x+sqrt(x^2+1)));
```

$$E2 := \frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)x^2 - \left(\int \frac{1}{2}\,\frac{\left(1+\frac{x}{\sqrt{x^2+1}}\right)x^2}{x+\sqrt{x^2+1}}\,\mathrm{d}x\right)$$

```
> simplify((1/2)*(1+x/sqrt(x^2+1))*x^2/(x+sqrt(x^2+1)));
```

$$\frac{1}{2}\,\frac{x^2}{\sqrt{x^2+1}}$$

```
> E21:=Int((1/2)*(1+x/sqrt(x^2+1))*x^2/(x+sqrt(x^2+1)), x);
```

$$E21 := \int \frac{1}{2}\,\frac{\left(1+\frac{x}{\sqrt{x^2+1}}\right)x^2}{x+\sqrt{x^2+1}}\,\mathrm{d}x$$

```
> E22:=intparts(E21,x);
```

$$E22 := \frac{1}{2}x\sqrt{x^2+1} - \left(\int \frac{1}{2}\sqrt{x^2+1}\,\mathrm{d}x\right)$$

```
> E23:=value(E22);
```

$$E23 := \frac{1}{4}x\sqrt{x^2+1} - \frac{1}{4}\operatorname{arcsinh}(x)$$

```
> E2new:=(1/2)*ln(x+sqrt(x^2+1))*x^2-E23;
```

$$E2new := \frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)x^2 - \frac{1}{4}x\sqrt{x^2+1} + \frac{1}{4}\operatorname{arcsinh}(x)$$

**Compute the integral $\int x^3 \arctan(x)\,\mathrm{d}x$. Verify your answer directly.** **Use partial fraction decomposition for the integral you get.**

```
> E1:=Int(x^3*arctan(x), x);
```

$$E1 := \int x^3 \arctan(x)\,\mathrm{d}x$$

```
> E2:=intparts(E1,arctan(x));
```

$$E2 := \frac{1}{4}x^4\arctan(x) - \left(\int \frac{1}{4}\,\frac{x^4}{x^2+1}\,\mathrm{d}x\right)$$

```
> E21:=Int((1/4)*x^4/(x^2+1), x);
```

$$E21 := \int \frac{1}{4} \frac{x^4}{x^2+1} \, dx$$

```
> E22:=intparts(E21,x);
```

$$E22 := x \left( \frac{1}{8} x^2 - \frac{1}{8} \ln(x^2+1) \right) - \left( \int \left( \frac{1}{8} x^2 - \frac{1}{8} \ln(x^2+1) \right) dx \right)$$

```
> E221:=Int((1/8)*ln(x^2+1),x);
```

$$E221 := \int \frac{1}{8} \ln(x^2+1) \, dx$$

```
> E222:=intparts(E221,ln(x^2+1));
```

$$E222 := \frac{1}{8} x \ln(x^2+1) - \left( \int \frac{1}{4} \frac{x^2}{x^2+1} \, dx \right)$$

```
> E22new:=x*((1/8)*x^2-(1/8)*ln(x^2+1))-(Int((1/8)*x^2, x))+E222;
```

$$E22new := x \left( \frac{1}{8} x^2 - \frac{1}{8} \ln(x^2+1) \right) - \left( \int \frac{1}{8} x^2 \, dx \right) + \frac{1}{8} x \ln(x^2+1) - \left( \int \frac{1}{4} \frac{x^2}{x^2+1} \, dx \right)$$

```
> E2new:=(1/4)*x^4*arctan(x)-E22new;
```

$$E2new := \frac{1}{4} x^4 \arctan(x) - x \left( \frac{1}{8} x^2 - \frac{1}{8} \ln(x^2+1) \right) + \int \frac{1}{8} x^2 \, dx - \frac{1}{8} x \ln(x^2+1) + \int \frac{1}{4} \frac{x^2}{x^2+1} \, dx$$

```
> value(E2new);
```

$$\frac{1}{4} x^4 \arctan(x) - x \left( \frac{1}{8} x^2 - \frac{1}{8} \ln(x^2+1) \right) + \frac{1}{24} x^3 - \frac{1}{8} x \ln(x^2+1) + \frac{1}{4} x - \frac{1}{4} \arctan(x) \tag{1}$$

```
>
```