MATH 156 Lab 11

> *BYUNG DO PARK*

Topic 1: Comparing integrals.

We recall that if $f(x) < g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$. This inequality allows us to estimate some integrals that are difficult to calculate.

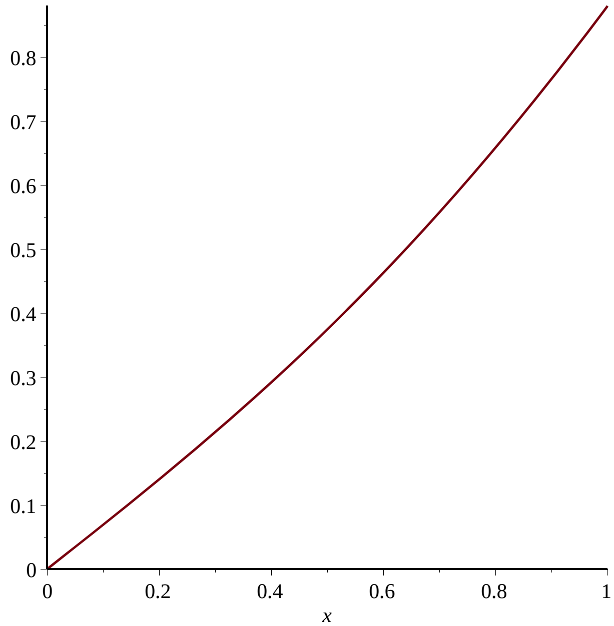
Show that $\int_0^1 x \ln\left(1+\sqrt{1+x^2}\right) dx < \ln\left(1+\sqrt{2}\right)$**.**

First we graph the function to integrate on the given interval.

```
> f:=x->x*ln(1+sqrt(1+x^2));
```

$$f := x \rightarrow x \ln\left(1+\sqrt{1+x^2}\right)$$

```
> with(plots):plot(f(x), x=0..1);
```



From the graph it is obvious that it is an increasing function. Consequently, $f(x) < f(1)$ for $x < 1$. Now we can take $g(x) = \ln(1+\sqrt{2})$, since $f(1) = \ln(1+\sqrt{2})$.

The integral $\int_0^1 g(x)\, dx$ is equal to $\ln(1+\sqrt{2})$, as the function is constant and the interval has length 1.

Show that $0.375 < \ln(1.5)$**. Here are the steps to follow: Recall that** $\ln(1.5) = \int_1^{1.5} \frac{1}{x}\, dx$**. Plot on the same graph the functions**

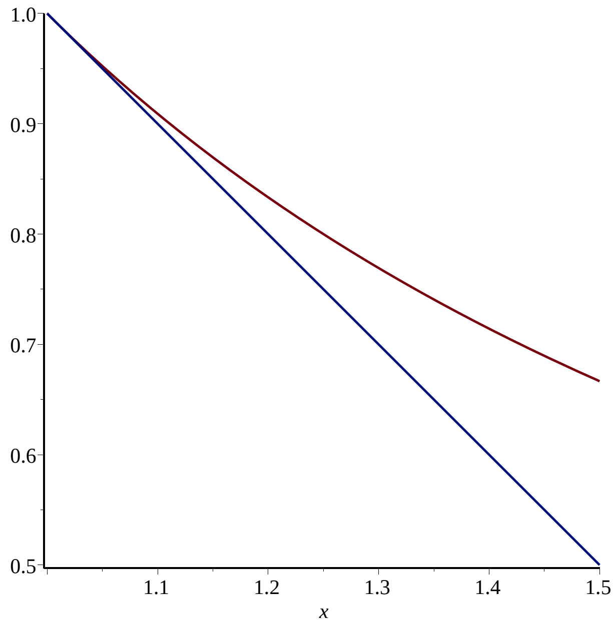
$g(x) = \frac{1}{x}$ **and** $f(x) = 2 - x$**. What do you notice? Explain why**

$$\int_{1}^{1.5} (2 - x)\, dx = 0.375.$$

```
> f:=x->1/x; g:=x->2-x; plot({f(x),g(x)},x=1..1.5);
```

$$f := x \rightarrow \frac{1}{x}$$

$$g := x \rightarrow 2 - x$$




```
> int(2-x, x = 1 .. 1.5);
```

$$0.3750000000$$

```
>
```

Topic 2: Improper integrals.

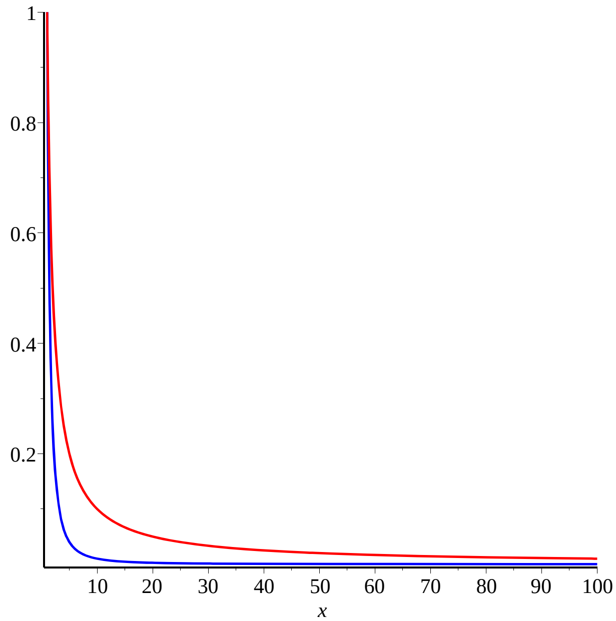
We plot first on the same graph the functions $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$. Just by looking at the graphs over a long interval, we cannot decide which improper integral $\int_1^{infinity} \frac{1}{x}\,dx$ or $\int_1^{infinity} \frac{1}{x^2}\,dx$ converges or diverges. However, it becomes clear that $\frac{1}{x^2} < \frac{1}{x}$ for all $1 < x$ and, consequently, the area below the graph of $\frac{1}{x}$ is larger than the area below the graph of $\frac{1}{x^2}$.

```
> reset:f:=x->1/x^2; g:=x->1/x;
```

$$f := x \to \frac{1}{x^2}$$

$$g := x \to \frac{1}{x}$$

```
> plot([f(x),g(x)], x=1..100, color=[blue, red]);
```



Putting the names of the functions in square brackets assigns this order to the plots. The color command assigns the color blue to the graph $f(x) = \frac{1}{x^2}$ and the color red to the graph $f(x) = \frac{1}{x^2}$. With Maple we can compute the improper integrals and see that $\int_1^{infinity} \frac{1}{x}\,dx$=*infinity*, while $\int_1^{infinity} \frac{1}{x^2}\,dx$=1.

```
> A:=Int(1/x,x = 1 .. infinity);
```

$$A := \int_1^{\infty} \frac{1}{x}\,dx$$

```
> value(A);
```

$$\infty$$

```
> B:=Int(1/x^2,x = 1 .. infinity);
```

$$B := \int_{1}^{\infty} \frac{1}{x^2}\, dx$$

```
> value(B);
```

$$1$$

To understand the convergence or divergence of these improper integrals, we compute values of the integrals $\int_{1}^{b} \frac{1}{x}\, dx$ and $\int_{1}^{b} \frac{1}{x^2}\, dx$ for various b. We can do this effectively with a loop.

```
> for j from 1 to 20 do b:=10^j: evalf(int(g(x), x=1..b)), evalf
  (int(f(x), x=1..b)): od;
```

$b := 10$

2.302585093, 0.9000000000

$b := 100$

4.605170185, 0.9900000000

$b := 1000$

6.907755278, 0.9990000000

$b := 10000$

9.210340370, 0.9999000000

$b := 100000$

11.51292546, 0.9999900000

$b := 1000000$

13.81551056, 0.9999990000

$b := 10000000$

16.11809564, 0.9999999000

$b := 100000000$

18.42068074, 0.9999999900

$b := 1000000000$

20.72326584, 0.9999999990

$b := 10000000000$

23.02585093, 0.9999999999

$b := 100000000000$

$$25.32843602,\ 1.000000000$$

$$b := 1000000000000$$

$$27.63102111,\ 1.000000000$$

$$b := 10000000000000$$

$$29.93360621,\ 1.000000000$$

$$b := 100000000000000$$

$$32.23619130,\ 1.000000000$$

$$b := 1000000000000000$$

$$34.53877639,\ 1.000000000$$

$$b := 10000000000000000$$

$$36.84136148,\ 1.000000000$$

$$b := 100000000000000000$$

$$39.14394657,\ 1.000000000$$

$$b := 1000000000000000000$$

$$41.44653167,\ 1.000000000$$

$$b := 10000000000000000000$$

$$43.74911676,\ 1.000000000$$

$$b := 100000000000000000000$$

$$46.05170185,\ 1.000000000$$

Make a table of values for the improper integrals $\int_2^{infinity} e^{-0.5\,x}\,dx$ **and** $\int_2^{infinity} e^{-0.1\,x}\,dx$**. Can you decide whether they converge or diverge?**

```
> int(exp(-0.5*x), x = 2..infinity); int(exp(-.1*x), x = 2..
  infinity);
```

$$0.7357588823$$

$$8.187307531$$

```
> for j from 1 to 20 do b:=10^j: evalf(int(exp(-0.5*x), x=2..b)),
  evalf(int(exp(-.1*x), x=2..b)): od;
```

$$b := 10$$

$$0.7222829883,\ 4.508513119$$

$$b := 100$$

$$0.7357588823,\ 8.186853531$$

$b := 1000$

0.7357588823, 8.187307531

$b := 10000$

0.7357588823, 8.187307531

$b := 100000$

0.7357588823, 8.187307531

$b := 1000000$

0.7357588823, 8.187307531

$b := 10000000$

0.7357588823, 8.187307531

$b := 100000000$

0.7357588823, 8.187307531

$b := 1000000000$

0.7357588823, 8.187307531

$b := 10000000000$

0.7357588823, 8.187307531

$b := 100000000000$

0.7357588823, 8.187307531

$b := 1000000000000$

0.7357588823, 8.187307531

$b := 10000000000000$

0.7357588823, 8.187307531

$b := 100000000000000$

0.7357588823, 8.187307531

$b := 1000000000000000$

0.7357588823, 8.187307531

$b := 10000000000000000$

0.7357588823, 8.187307531

$b := 100000000000000000$

0.7357588823, 8.187307531

$b := 1000000000000000000$

0.7357588823, 8.187307531

$b := 10000000000000000000$

0.7357588823, 8.187307531

$b := 100000000000000000000$

0.7357588823, 8.187307531

They both seem to converge. In fact it seems that $\int_2^{infinity} e^{-0.5x}\,dx = 0.7357588823$ and $\int_2^{infinity} e^{-0.1x}\,dx = 8.187307531$. **Evaluate the integrals by hand and show that these answers are true.**

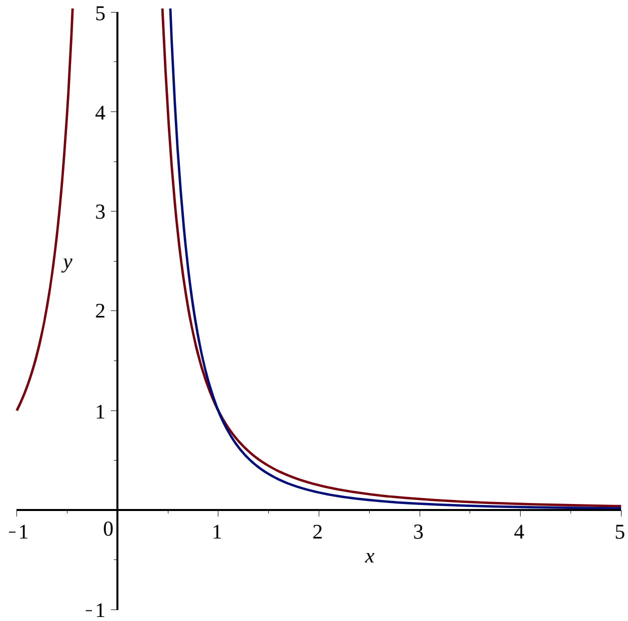
Topic 3: Comparing improper integrals.

Plot on the same graph $f(x) = \frac{1}{x^{2.5}}$ **and** $g(x) = \frac{1}{x^2}$**. Explain why the improper integral** $\int_1^{infinity} \frac{1}{x^{2.5}}\,dx$ **converges.**

```
> f:=x->1/x^(2.5); g:=x->1/x^(2); plot({f(x),g(x)}, x=-1..5, y=-1.
  .5);
```

$$f := x \rightarrow \frac{1}{x^{2.5}}$$

$$g := x \rightarrow \frac{1}{x^2}$$



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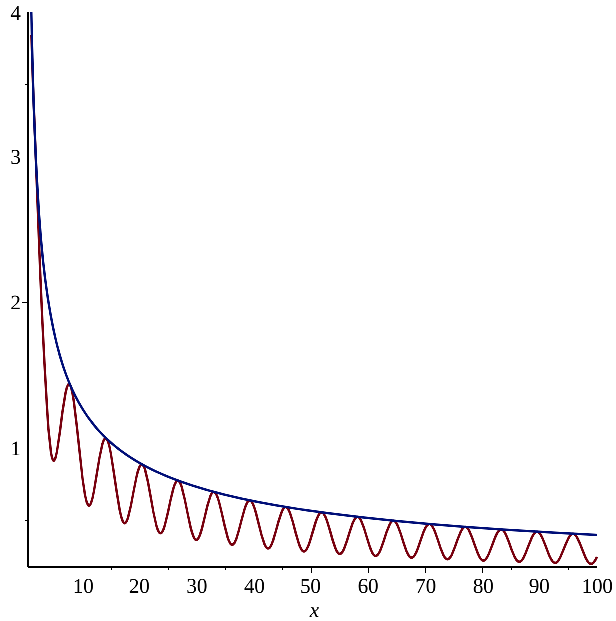
Decide whether the improper integral $\int_1^{\infty} \frac{\sin(x) + 3}{\sqrt{x}}\,\mathrm{d}x$ **converges or not.**

```
> plot({(sin(x)+3)/(sqrt(x)),4/(sqrt(x))},x=1..100);
```



$$-2$$

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