MATH 156 LAB 12

BYUNG DO PARK

Topic 1: Sequences and their limits.

We can define a sequence given by an explicit formula $a_n = f(n)$ by defining the function $f(x)$. Example: The sequence $a_n = 16 - 16\left(\frac{1}{2}\right)^n$.

```
> a:=n->16-16*(1/2)^n;
```

$$a := n \to 16 - 16\left(\frac{1}{2}\right)^n$$

```
> a(1);a(2);a(3);
```

8

12

14

We can easily make a list of its values with a loop command.

```
> for n from 1 to 20 do evalf(a(n));od;
```

8.

12.

14.

15.

15.50000000

15.75000000

15.87500000

15.93750000

15.96875000

15.98437500

15.99218750

15.99609375

15.99804688

15.99902344

15.99951172

15.99975586

15.99987793

15.99993896

$$15.99996948$$

$$15.99998474$$

We see that the numbers get closer and closer to 16. In fact $\lim_{n} a_n = 16$. We can compute this limit with the Limit command of Maple:

```
> Limit(a(m), m=infinity);
```

$$\lim_{m \to \infty} \left(16 - 16 \left(\frac{1}{2}\right)^m\right)$$

```
> value(%);
```

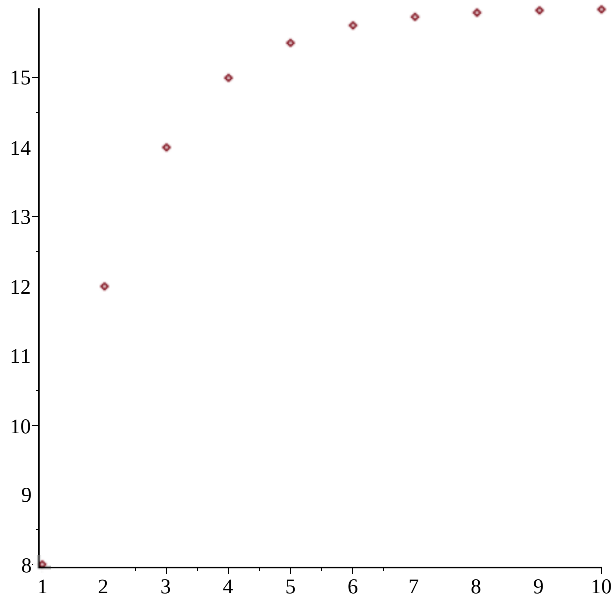
$$16$$

Graphically we can see the sequence by creating a list of the points $[n, a_n]$:

```
> graph:=[seq([n,a(n)], n=1..10)];
```

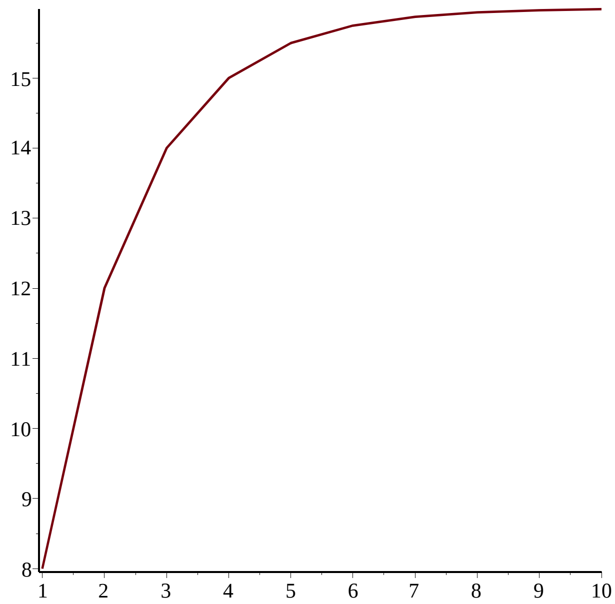
$$graph := \left[[1, 8], [2, 12], [3, 14], [4, 15], \left[5, \frac{31}{2}\right], \left[6, \frac{63}{4}\right], \left[7, \frac{127}{8}\right], \left[8, \frac{255}{16}\right], \left[9, \frac{511}{32}\right], \left[10, \frac{1023}{64}\right]\right]$$

```
> plot(graph, style=point);
```



The style=point option plots a dot or star at the corresponding point. We can also use the option style=line. This produces:

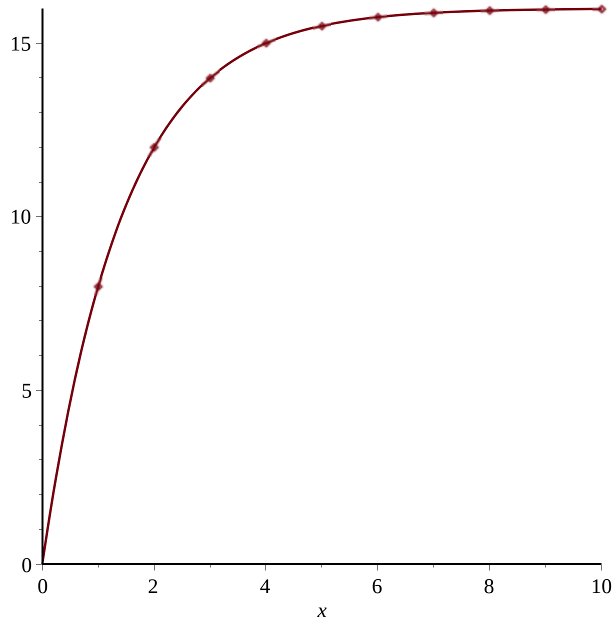
```
> plot(graph, style=line);
```



We see that the segments joining the points become eventually almost horizontal at height 16. This is the limit of the sequence. We can also plot the function and see the points of the sequence on it.

```
> gr1:=plot(graph, style=point):
> gr2:=plot(a(x), x=0..10):
> with(plots); display(gr1, gr2);
```

[*animate*, *animate3d*, *animatecurve*, *arrow*, *changecoords*, *complexplot*, *complexplot3d*, *conformal*, *conformal3d*, *contourplot*, *contourplot3d*, *coordplot*, *coordplot3d*, *densityplot*, *display*, *dualaxisplot*, *fieldplot*, *fieldplot3d*, *gradplot*, *gradplot3d*, *implicitplot*, *implicitplot3d*, *inequal*, *interactive*, *interactiveparams*, *intersectplot*, *listcontplot*, *listcontplot3d*, *listdensityplot*, *listplot*, *listplot3d*, *loglogplot*, *logplot*, *matrixplot*, *multiple*, *odeplot*, *pareto*, *plotcompare*, *pointplot*, *pointplot3d*, *polarplot*, *polygonplot*, *polygonplot3d*, *polyhedra_supported*, *polyhedraplot*, *rootlocus*, *semilogplot*, *setcolors*, *setoptions*, *setoptions3d*, *spacecurve*, *sparsematrixplot*, *surfdata*, *textplot*, *textplot3d*, *tubeplot*]



Investigate the limit of the sequence b_n =%?$\sqrt[n]{2}$. **Make a table showing the terms with** n **a multiple of 10 and show the sequence graphically.**

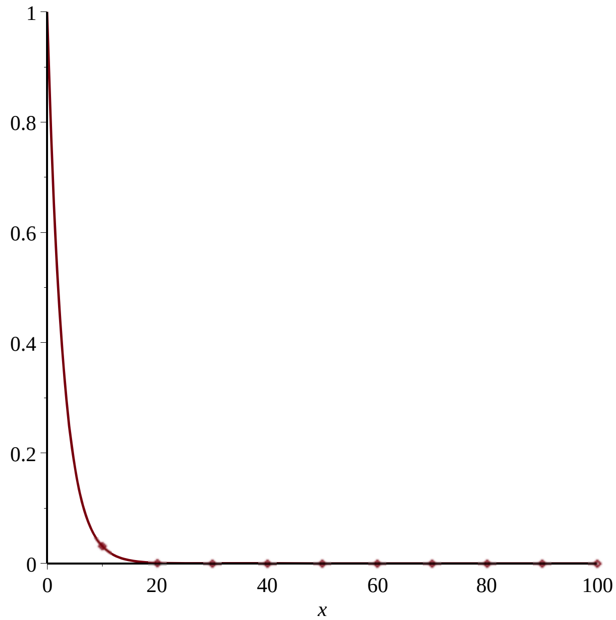
```
> b:=n->(1/2^n)*(2)^(n/2); limit(b(n), n=infinity); graph:=[seq(
  [10*n,b(10*n)], n=1..10)];
```

$$b := n \rightarrow \frac{2^{\frac{1}{2}n}}{2^n}$$

$$0$$

$$graph := \left[\left[10, \frac{1}{32}\right], \left[20, \frac{1}{1024}\right], \left[30, \frac{1}{32768}\right], \left[40, \frac{1}{1048576}\right], \left[50, \frac{1}{33554432}\right], \left[60, \frac{1}{1073741824}\right], \left[70, \frac{1}{34359738368}\right], \left[80, \frac{1}{1099511627776}\right], \left[90, \frac{1}{35184372088832}\right], \left[100, \frac{1}{1125899906842624}\right]\right]$$

```
> gr1:=plot(graph, style=point):
> gr2:=plot(b(x), x=0..100):
> with(plots): display(gr1, gr2);
```



It seems that the limit is 0.

Investigate the sequence $c_n = \left(-\frac{1}{3}\right)^n$. Find its limit, make a table and show the first 10 terms of the sequence graphically.

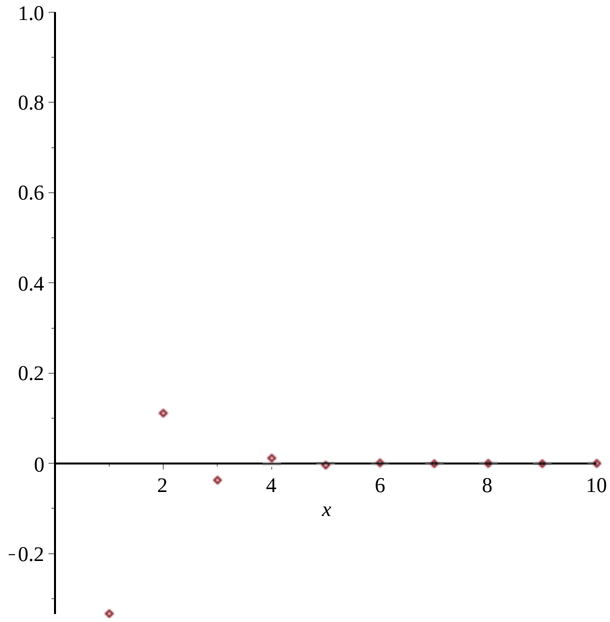
```
> c:=n->(-1/3)^(n); limit(c(n), n=infinity); graph:=[seq([n,c(n)], n=1..10)];
```

$$c := n \rightarrow \left(-\frac{1}{3}\right)^n$$

$$0$$

$$graph := \left[\left[1, -\frac{1}{3}\right], \left[2, \frac{1}{9}\right], \left[3, -\frac{1}{27}\right], \left[4, \frac{1}{81}\right], \left[5, -\frac{1}{243}\right], \left[6, \frac{1}{729}\right], \left[7, -\frac{1}{2187}\right], \left[8, \frac{1}{6561}\right], \left[9, -\frac{1}{19683}\right], \left[10, \frac{1}{59049}\right]\right]$$

```
> gr1:=plot(graph, style=point):
> gr2:=plot(c(x), x=0..10):
> with(plots): display(gr1, gr2);
```




```
> Notice that the base is negative.
```

Investigate the sequence $d_n = 1.7^n$. Find its limit, make a table and show the first 10 terms of the sequence graphically.

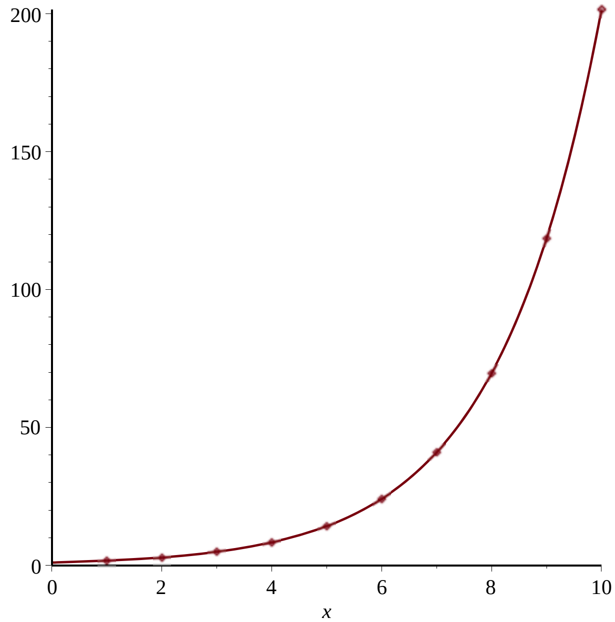
```
> d:=n->(1.7)^(n); limit(d(n), n=infinity); graph:=[seq([n,d(n)], n=1..10)];
```

$$d := n \rightarrow 1.7^n$$

$$Float(\infty)$$

$graph := [[1, 1.7], [2, 2.89], [3, 4.913], [4, 8.3521], [5, 14.19857], [6, 24.137569], [7, 41.0338673], [8, 69.75757441], [9, 118.5878765], [10, 201.5993900]]$

```
> gr1:=plot(graph, style=point):
> gr2:=plot(d(x), x=0..10):
> with(plots): display(gr1, gr2);
```



Example: The Fibonacci sequence. Compute the first 50 terms of the sequence given by $f(1) = 1$, $f(2) = 1$ and the recursion

$f(n) := f(n-1) + f(n-2)$

```
> f[1]:=1; f[2]:=1;
```

$$f_1 := 1$$

$$f_2 := 1$$

```
> for n from 3 to 50 do f[n]:=f[n-1]+f[n-2] od;
```

$$f_3 := 2$$

$$f_4 := 3$$

$$f_5 := 5$$

$$f_6 := 8$$

$$f_7 := 13$$

$$f_8 := 21$$

$$f_9 := 34$$

$$f_{10} := 55$$

$$f_{11} := 89$$

$$f_{12} := 144$$

$$f_{13} := 233$$

$$f_{14} := 377$$

$$f_{15} := 610$$

$$f_{16} := 987$$

$$f_{17} := 1597$$

$$f_{18} := 2584$$

$$f_{19} := 4181$$

$$f_{20} := 6765$$

$$f_{21} := 10946$$

$$f_{22} := 17711$$

$$f_{23} := 28657$$

$$f_{24} := 46368$$

$$f_{25} := 75025$$

$$f_{26} := 121393$$
$$f_{27} := 196418$$
$$f_{28} := 317811$$
$$f_{29} := 514229$$
$$f_{30} := 832040$$
$$f_{31} := 1346269$$
$$f_{32} := 2178309$$
$$f_{33} := 3524578$$
$$f_{34} := 5702887$$
$$f_{35} := 9227465$$
$$f_{36} := 14930352$$
$$f_{37} := 24157817$$
$$f_{38} := 39088169$$
$$f_{39} := 63245986$$
$$f_{40} := 102334155$$
$$f_{41} := 165580141$$
$$f_{42} := 267914296$$
$$f_{43} := 433494437$$
$$f_{44} := 701408733$$
$$f_{45} := 1134903170$$
$$f_{46} := 1836311903$$
$$f_{47} := 2971215073$$
$$f_{48} := 4807526976$$
$$f_{49} := 7778742049$$
$$f_{50} := 12586269025$$

The easiest way to work with sequences defined recursively is to use a loop.

Topic 2: Infinite series and their sums.

Using the sum command we can find the partial sums of various series and then their sum: Example: The series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Its partial sums s_N are given by $s_N = \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n$. We can use a loop to calculate s_N for $N = 1 .. 30$.

```
> restart;
> s:=N->sum((1/2)^n,n = 1 .. N);
```

$$s := N \to \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n$$

```
> for N from 1 to 30 do evalf(s(N));od;
```

0.5000000000

0.7500000000

0.8750000000

0.9375000000

0.9687500000

0.9843750000

0.9921875000

0.9960937500

0.9980468750

0.9990234375

0.9995117188

0.9997558594

0.9998779297

0.9999389648

0.9999694824

0.9999847412

0.9999923706

0.9999961853

0.9999980927

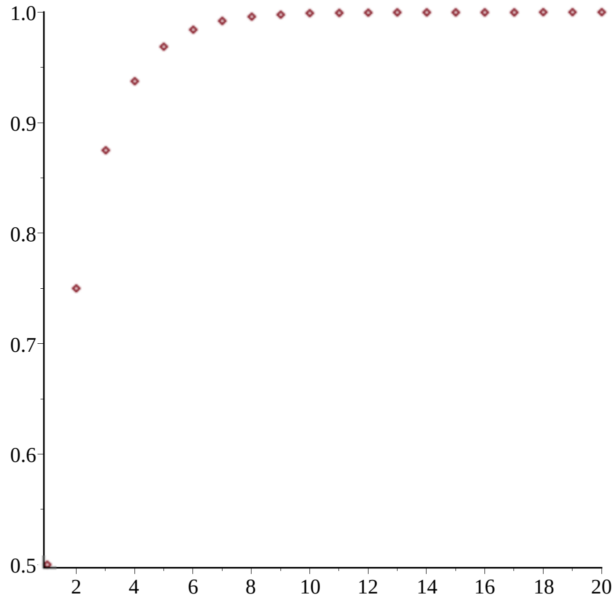
0.9999990463
0.9999995232
0.9999997616
0.9999998808
0.9999999404
0.9999999702
0.9999999851
0.9999999925
0.9999999963
0.9999999981
0.9999999991

It seems that the limit of the partial sums is 1. This is the sum of the infinite series. Here is the graph of the partial sums:

```
> graph:=[seq([m, s(m)], m=1..20)];
```

$$graph := \left[\left[1, \frac{1}{2}\right], \left[2, \frac{3}{4}\right], \left[3, \frac{7}{8}\right], \left[4, \frac{15}{16}\right], \left[5, \frac{31}{32}\right], \left[6, \frac{63}{64}\right], \left[7, \frac{127}{128}\right], \left[8, \frac{255}{256}\right], \left[9, \frac{511}{512}\right], \left[10, \frac{1023}{1024}\right], \left[11, \frac{2047}{2048}\right], \left[12, \frac{4095}{4096}\right], \left[13, \frac{8191}{8192}\right], \left[14, \frac{16383}{16384}\right], \left[15, \frac{32767}{32768}\right], \left[16, \frac{65535}{65536}\right], \left[17, \frac{131071}{131072}\right], \left[18, \frac{262143}{262144}\right], \left[19, \frac{524287}{524288}\right], \left[20, \frac{1048575}{1048576}\right]\right]$$

```
> plot(graph, style=point);
```



We can ask Maple whether it can calculate a formula for the partial sums. The answer is yes.

```
> restart;s(N):=sum((1/2)^n, n=1..N);
```

$$s(N) := -2\left(\frac{1}{2}\right)^{N+1} + 1$$

```
> Limit(s(N), N=infinity);
```

$$\lim_{N\to\infty}\left(-2\left(\frac{1}{2}\right)^{N+1} + 1\right)$$

```
> value(%);
```

$$1$$

In fact we saw the formula $\sum_{n=0}^{N} a\,r^n = \frac{a\left(1 - r^{N+1}\right)}{1 - r}$ in class. **Verify the formula for $N = 1$ .. 20 with $a = 5$ and**

$$r = \frac{2}{3}.$$

```
> for k from 1 to 2 do N:=k; sum(5*(2/3)^n, n=0..N); 5*(1-(2/3)^
  (N+1))/(1-(2/3)); od;
```

$$N := 1$$

$$\frac{25}{3}$$

$$\frac{25}{3}$$

$$N := 2$$

$$\frac{95}{9}$$

$$\frac{95}{9}$$

Explain why $\sum_{n=0}^{infinity} 5 \left(\frac{2}{3}\right)^n$ **=15.**

```
> s:=N->sum(5*(2/3)^n ,n=0..N);
```

$$s := N \to \sum_{n=0}^{N} 5 \left(\frac{2}{3}\right)^n$$

```
> limit(s(x),x=infinity);
```

$$15$$

Compute the sum of the series $\sum_{n=1}^{infinity} \frac{1}{n\,(n+2)}$**. This is a telescoping series. To see where its sum comes from, compute the partial sums** S_N **and use partial fractions to** $\frac{1}{n\,(n+2)}$ **to see the cancellation.**

```
> s:=x->sum(1/(n*(n+2)),n=1..x); limit(s(x),x=infinity);
```

$$s := x \to \sum_{n=1}^{x} \frac{1}{n\,(n+2)}$$

$\frac{3}{4}$