MATH 156 LAB 14

BYUNG DO PARK

Topic 1: Taylor polynomials.

We have seen in Calculus 1 that the tangent line is the best linear approximation to the graph of a function. Let us take the function $f(x) = \ln(x)$ and look at the tangent line and the graph close to the point (1,0).

```
> f:=x->ln(x);
```

$$f := x \to \ln(x)$$

```
> D(f)(1);
```

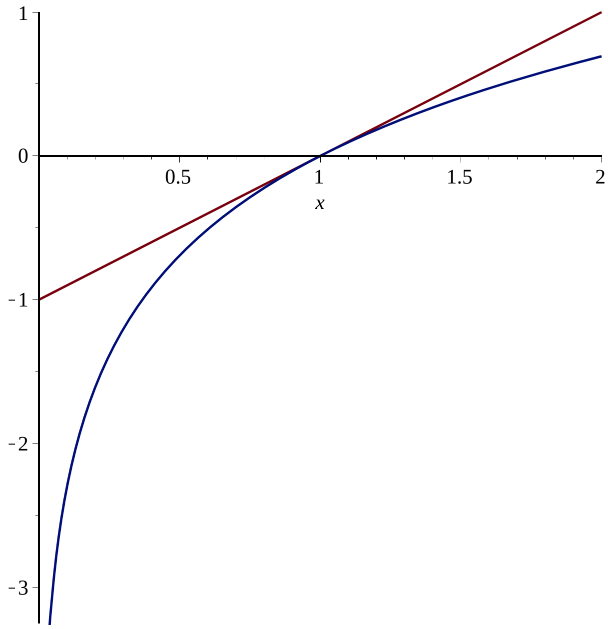
$$1$$

Since the slope of the tangent line is 1 and the point of contact is (1, 0) the equation of the tangent line is $y = x - 1$.

```
> g:=x->x-1;
```

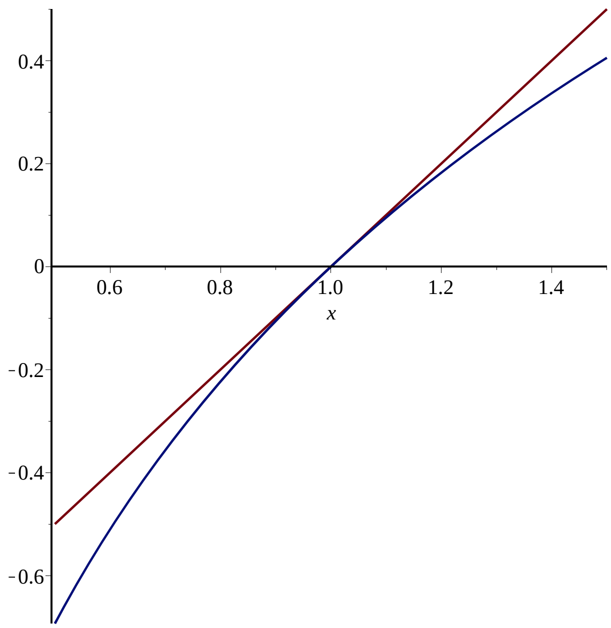
$$g := x \to x - 1$$

```
> plot({f(x), g(x)}, x=0..2);
```

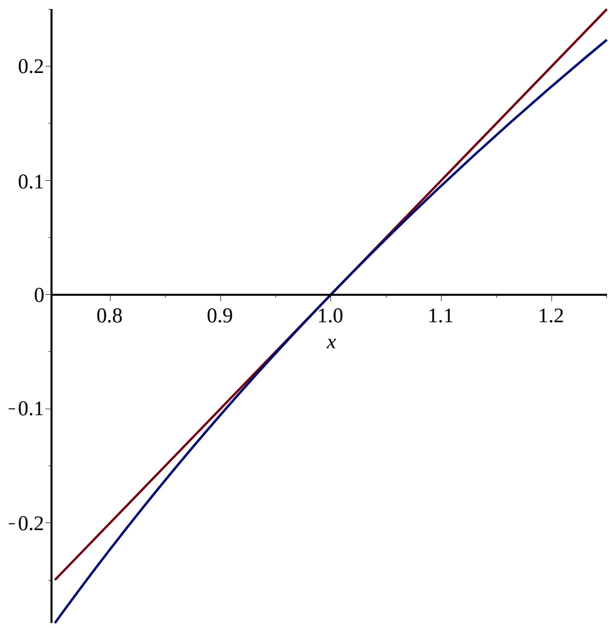


We zoom further by choosing the x-range to be smaller.

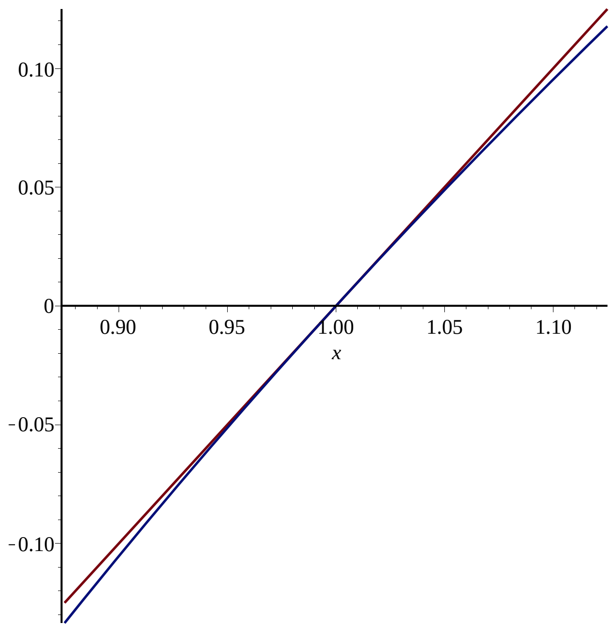
```
> plot({f(x), g(x)}, x=0.5..1.5);
```




```
> plot({f(x), g(x)}, x=0.75..1.25);
```




```
> plot({f(x), g(x)}, x=0.875..1.125);
```



We see that the closer we look at (1,0), the closer the tangent line is to the graph. Moreover, we notice the following. At every step we halved the size of the *x*-interval. The distance between the graph of $f(x) = \ln(x)$ and the tangent line is not only halved but gets smaller more rapidly than that. This is the meaning of the tangent line approximation. We can see the numerics as well. The closer we are to 1, the closer the value of the tangent line to the actual value of the function:

```
> for k from 1 to 10 do; x:= 1+2^(-k); actualvalue:=evalf(f(x));
  tangentapprox:=evalf(x-1); od;
```

$$x := \frac{3}{2}$$

$$actualvalue := 0.4054651081$$

$$tangentapprox := 0.5000000000$$

$$x := \frac{5}{4}$$

$$actualvalue := 0.2231435513$$

$$tangentapprox := 0.2500000000$$

$$x := \frac{9}{8}$$

$$actualvalue := 0.1177830357$$

$$tangentapprox := 0.1250000000$$

$$x := \frac{17}{16}$$

$$actualvalue := 0.06062462182$$

$$tangentapprox := 0.06250000000$$

$$x := \frac{33}{32}$$

$$actualvalue := 0.03077165867$$

$$tangentapprox := 0.03125000000$$

$$x := \frac{65}{64}$$

$$actualvalue := 0.01550418654$$

$$tangentapprox := 0.01562500000$$

$$x := \frac{129}{128}$$

$$actualvalue := 0.007782140442$$

$$tangentapprox := 0.007812500000$$

$$x := \frac{257}{256}$$

$$actualvalue := 0.003898640416$$

$$tangentapprox := 0.003906250000$$

$$x := \frac{513}{512}$$

$$actualvalue := 0.001951220131$$

$$tangentapprox := 0.001953125000$$

$$x := \frac{1025}{1024}$$

$$actualvalue := 0.0009760854735$$

$$tangentapprox := 0.0009765625000$$

We can measure the error in the tangent line approximation by computing $\ln(x) - (x-1)$.

```
> for k from 1 to 10 do; x:= 1+2^(-k); evalf(f(x)-(x-1)); od;
```

$$x := \frac{3}{2}$$

$$-0.0945348919$$

$$x := \frac{5}{4}$$

$$-0.0268564487$$

$$x := \frac{9}{8}$$

$$-0.0072169643$$

$$x := \frac{17}{16}$$

$$-0.00187537818$$

$$x := \frac{33}{32}$$

$$-0.00047834133$$

$$x := \frac{65}{64}$$

$$-0.00012081346$$

$$x := \frac{129}{128}$$

$$-0.000030359558$$

$$x := \frac{257}{256}$$

$$-0.000007609584$$

$$x := \frac{513}{512}$$

$$-0.000001904869$$

$$x := \frac{1025}{1024}$$

$$-4.770265\ 10^{-7}$$

The errors get smaller the closer we are to $x = 1$. The errors are negative, because the tangent line overestimates the function. The tangent line lies above the graph of the function, since the function is concave downwards. **Make similar tables for values of** $x < 1$.

```
> for k from 1 to 10 do; x:= 1-2^(-k); actualvalue:=evalf(f(x));
  tangentapprox:=evalf(x-1); od;
> for k from 1 to 10 do; x:= 1-2^(-k); evalf(f(x)-(x-1)); od;
```

$$x := \frac{1}{2}$$

$$actualvalue := -0.693147180559945$$

$$tangentapprox := -0.500000000000000$$

$$x := \frac{3}{4}$$

$$actualvalue := -0.287682072451781$$

$$tangentapprox := -0.250000000000000$$

$$x := \frac{7}{8}$$

$$actualvalue := -0.133531392624523$$

$$tangentapprox := -0.125000000000000$$

$$x := \frac{15}{16}$$

$$actualvalue := -0.0645385211375712$$

$$tangentapprox := -0.0625000000000000$$

$$x := \frac{31}{32}$$

$$actualvalue := -0.0317486983145803$$

$$tangentapprox := -0.0312500000000000$$

$$x := \frac{63}{64}$$

$$actualvalue := -0.0157483569681392$$

$$tangentapprox := -0.0156250000000000$$

$$x := \frac{127}{128}$$

$$actualvalue := -0.00784317746102589$$

$$tangentapprox := -0.00781250000000000$$

$$x := \frac{255}{256}$$

$$actualvalue := -0.00391389932113633$$

$$tangentapprox := -0.00390625000000000$$

$$x := \frac{511}{512}$$

$$actualvalue := -0.00195503483580335$$

$$tangentapprox := -0.00195312500000000$$

$$x := \frac{1023}{1024}$$

$$actualvalue := -0.000977039647826613$$

$$tangentapprox := -0.000976562500000000$$

$$x := \frac{1}{2}$$

$$-0.193147180559945$$

$$x := \frac{3}{4}$$

$$-0.037682072451781$$

$$x := \frac{7}{8}$$

$$-0.008531392624523$$

$$x := \frac{15}{16}$$

$$-0.0020385211375712$$

$$x := \frac{31}{32}$$

$$-0.0004986983145803$$

$$x := \frac{63}{64}$$

$$-0.0001233569681392$$

$$x := \frac{127}{128}$$

$$-0.00003067746102589$$

$$x := \frac{255}{256}$$

$$-0.00000764932113633$$

$$x := \frac{511}{512}$$

$$-0.00000190983580335$$

$$x := \frac{1023}{1024}$$

$$-4.77147826613\ 10^{-7}$$

On the other hand, if we are far away from (1,0), the tangent line approximation is not very good:

```
> evalf(f(2)); g(2);
```

0.693147180559945

1

```
> evalf(f(3)); g(3);
```

1.09861228866811

2

```
> evalf(f(4));g(4);
```

1.38629436111989

3

The reason is that the function $\ln(x)$ bends, as a concave downwards function, while the tangent line does not. Somehow we have to take into account the concavity. In Calculus 1 we saw that the concavity is measured by the second derivative.

```
> D(D(f))(1);
```

-1

The function $h(x) = x - 1 - \frac{(x-1)^2}{2}$ has the following properties: h(1)=f(1), h'(1)=f'(1) and h''(1)=f''(1):

```
> h:=x->(x-1)-(x-1)^2/2;
```

$$h := x \to x - 1 - \frac{1}{2}(x-1)^2$$

```
> h(1);f(1);
```

0

0

```
> D(h)(1); D(f)(1);
```

1

1

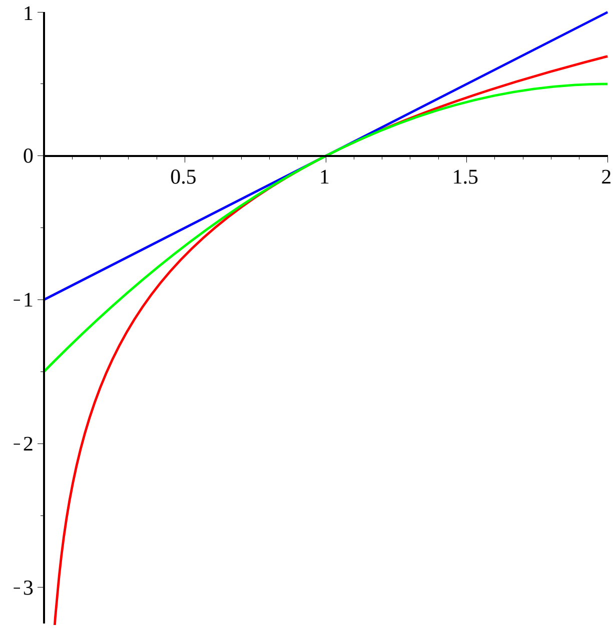
```
> D(D(h))(1);D(D(f))(1);
```

-1

-1

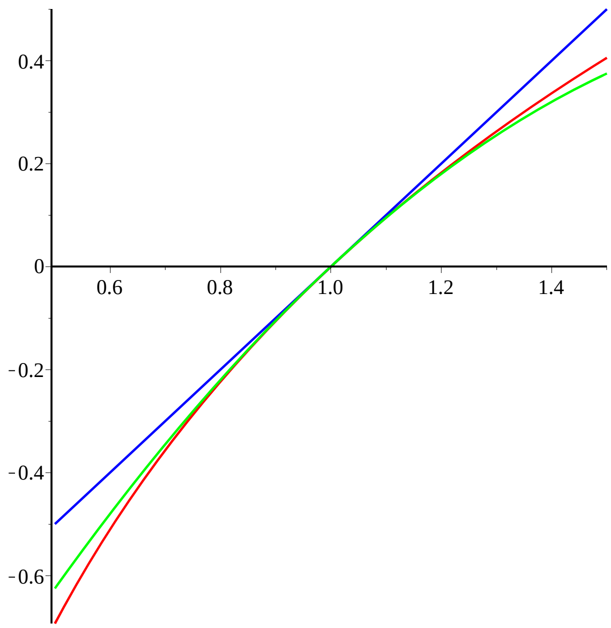
We now graph all the three functions.

```
> plot([f,g,h], 0..2, color=[red, blue, green]);
```

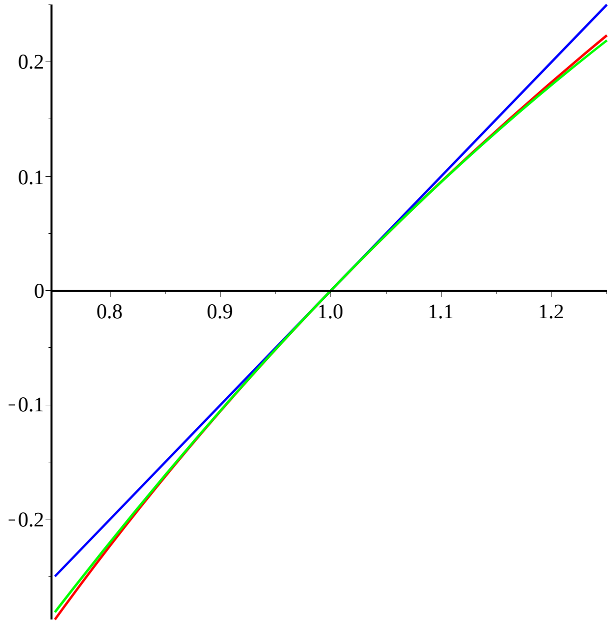


We painted the logarithmic function with red, the tangent line with blue and the function h(x) with green. We see that the quadratic function $h(x)$ seems to be closer to the graph of the logarithm than the tangent line. We zoom in by adjusting the range.

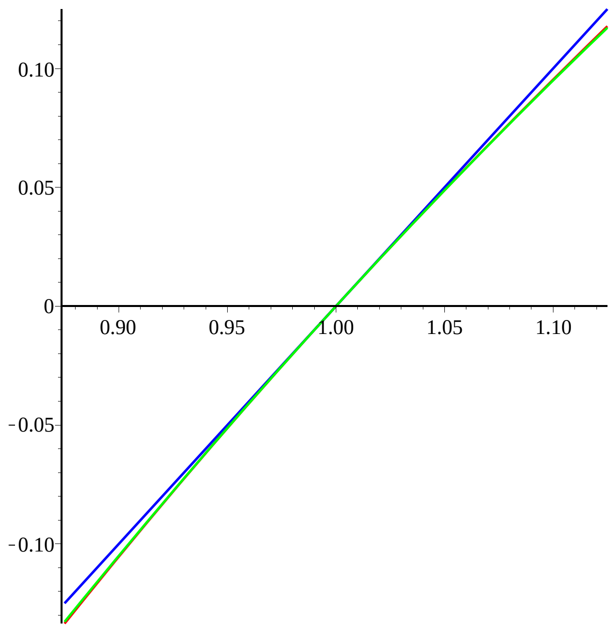
```
> plot([f,g,h], 0.5..1.5, color=[red, blue, green]);
```




```
> plot([f,g,h], 0.75..1.25, color=[red, blue, green]);
```




```
> plot([f,g,h], 0.875..1.125, color=[red, blue, green]);
```



At this moment we can hardly distinguish the quadratic function, which is a parabola, from the logarithm. We make also a table of values of all 3 functions.

```
> Digits:=15; for k from 1 to 10 do; x:= 1+2^(-k); actual:=evalf(f
  (x)); tangent:=evalf(g(x));parab:=evalf(h(x)) od;
```

$$Digits := 15$$

$$x := \frac{3}{2}$$

$$actual := 0.405465108108164$$

$$tangent := 0.500000000000000$$

$$parab := 0.375000000000000$$

$$x := \frac{5}{4}$$

$$actual := 0.223143551314210$$

$$tangent := 0.250000000000000$$

$$parab := 0.218750000000000$$

$$x := \frac{9}{8}$$

$$actual := 0.117783035656383$$

$$tangent := 0.125000000000000$$

$$parab := 0.117187500000000$$

$$x := \frac{17}{16}$$

$$actual := 0.0606246218164348$$

$$tangent := 0.0625000000000000$$

$$parab := 0.0605468750000000$$

$$x := \frac{33}{32}$$

$$actual := 0.0307716586667537$$

$$tangent := 0.0312500000000000$$

$$parab := 0.0307617187500000$$

$$x := \frac{65}{64}$$

$$actual := 0.0155041865359653$$

$$tangent := 0.0156250000000000$$

$$parab := 0.0155029296875000$$

$$x := \frac{129}{128}$$

$$actual := 0.00778214044205495$$

$$tangent := 0.00781250000000000$$

$$parab := 0.00778198242187500$$

$$x := \frac{257}{256}$$

$$actual := 0.00389864041565732$$

$$tangent := 0.00390625000000000$$

$$parab := 0.00389862060546875$$

$$x := \frac{513}{512}$$

$$actual := 0.00195122013126175$$

$$tangent := 0.00195312500000000$$

$$parab := 0.00195121765136719$$

$$x := \frac{1025}{1024}$$

$$actual := 0.000976085973055459$$

$$tangent := 0.000976562500000000$$

$$parab := 0.000976085662841797$$

We see that the values of $h(x)$ are closer to the logarithm than the corresponding values of the tangent line ($g(x)$). To measure the closeness we make a table of the errors: $\ln(x) - (x-1)$ and $\ln(x) - \left(x - 1 - \frac{(x-1)^2}{2}\right)$:

```
> for k from 1 to 10 do; x:= 1+2^(-k); errorfortangent:=evalf(f(x)-g(x));errorforparab:=evalf(f(x)-h(x)); od;
```

$$x := \frac{3}{2}$$

$$errorfortangent := -0.094534891891836$$

$$errorforparab := 0.030465108108164$$

$$x := \frac{5}{4}$$

$$errorfortangent := -0.026856448685790$$

$$errorforparab := 0.004393551314210$$

$$x := \frac{9}{8}$$

$$errorfortangent := -0.007216964343617$$

$$errorforparab := 0.000595535656383$$

$$x := \frac{17}{16}$$

$$errorfortangent := -0.0018753781835652$$

$$errorforparab := 0.0000777468164348$$

$$x := \frac{33}{32}$$

$$errorfortangent := -0.0004783413332463$$

$$errorforparab := 0.0000099399167537$$

$$x := \frac{65}{64}$$

$$errorfortangent := -0.0001208134640347$$

$$errorforparab := 0.0000012568484653$$

$$x := \frac{129}{128}$$

$$errorfortangent := -0.00003035955794505$$

$$errorforparab := 1.5802017995\ 10^{-7}$$

$$x := \frac{257}{256}$$

$$errorfortangent := -0.00000760958434268$$

$$errorforparab := 1.981018857\ 10^{-8}$$

$$x := \frac{513}{512}$$

$$errorfortangent := -0.00000190486873825$$

$$errorforparab := 2.47989456\ 10^{-9}$$

$$x := \frac{1025}{1024}$$

$$errorfortangent := -4.76526944541\ 10^{-7}$$

$$errorforparab := 3.10213662\ 10^{-10}$$

We see that the errors are smaller for the quadratic polynomial. The errors are positive for $h(x)$ because for $1 < x$ the parabola was below the graph of the logarithm. **Make a table of f, g, h and the errors for values of** $x < 1$.

```
> Digits:=15; for k from 1 to 10 do; x:= 1-2^(-k); actual:=evalf(f
  (x)); tangent:=evalf(g(x));parab:=evalf(h(x)) od;
```

$$Digits := 15$$

$$x := \frac{1}{2}$$

$$actual := -0.693147180559945$$

$$tangent := -0.500000000000000$$

$$parab := -0.625000000000000$$

$$x := \frac{3}{4}$$

$$actual := -0.287682072451781$$

$$tangent := -0.250000000000000$$

$$parab := -0.281250000000000$$

$$x := \frac{7}{8}$$

$$actual := -0.133531392624523$$

$$tangent := -0.125000000000000$$

$$parab := -0.132812500000000$$

$$x := \frac{15}{16}$$

$$actual := -0.0645385211375712$$

$$tangent := -0.0625000000000000$$

$$parab := -0.0644531250000000$$

$$x := \frac{31}{32}$$

$$actual := -0.0317486983145803$$

$$tangent := -0.0312500000000000$$

$$parab := -0.0317382812500000$$

$$x := \frac{63}{64}$$

$$actual := -0.0157483569681392$$

$$tangent := -0.0156250000000000$$

$$parab := -0.0157470703125000$$

$$x := \frac{127}{128}$$

$$actual := -0.00784317746102589$$

$$tangent := -0.00781250000000000$$

$$parab := -0.00784301757812500$$

$$x := \frac{255}{256}$$

$$actual := -0.00391389932113633$$

$$tangent := -0.00390625000000000$$

$$parab := -0.00391387939453125$$

$$x := \frac{511}{512}$$

$$actual := -0.00195503483580335$$

$$tangent := -0.00195312500000000$$

$$parab := -0.00195503234863281$$

$$x := \frac{1023}{1024}$$

$$actual := -0.000977039647826613$$

$$tangent := -0.000976562500000000$$

$$parab := -0.000977039337158203$$

```
> for k from 1 to 10 do; x:= 1-2^(-k); errorfortangent:=evalf(f(x)-
  g(x));errorforparab:=evalf(f(x)-h(x)); od;
```

$$x := \frac{1}{2}$$

$$errorfortangent := -0.193147180559945$$

$$errorforparab := -0.068147180559945$$

$$x := \frac{3}{4}$$

$$errorfortangent := -0.037682072451781$$

$$errorforparab := -0.006432072451781$$

$$x := \frac{7}{8}$$

$$errorfortangent := -0.008531392624523$$

$$errorforparab := -0.000718892624523$$

$$x := \frac{15}{16}$$

$$errorfortangent := -0.0020385211375712$$

$$errorforparab := -0.0000853961375712$$

$$x := \frac{31}{32}$$

$$errorfortangent := -0.0004986983145803$$

$$errorforparab := -0.0000104170645803$$

$$x := \frac{63}{64}$$

$$errorfortangent := -0.0001233569681392$$

$$errorforparab := -0.0000012866556392$$

$$x := \frac{127}{128}$$

$$errorfortangent := -0.00003067746102589$$

$$errorforparab := -1.5988290089\ 10^{-7}$$

$$x := \frac{255}{256}$$

$$errorfortangent := -0.00000764932113633$$

$$errorforparab := -1.992660508\ 10^{-8}$$

$$x := \frac{511}{512}$$

$$errorfortangent := -0.00000190983580335$$

$$errorforparab := -2.48717054\ 10^{-9}$$

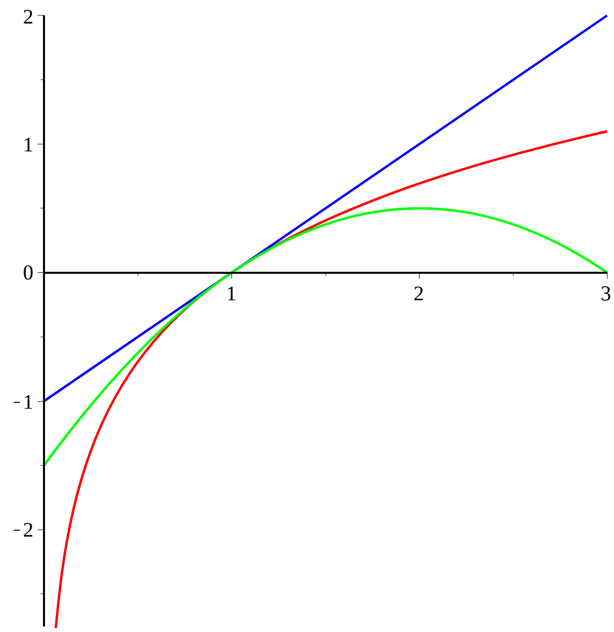
$$x := \frac{1023}{1024}$$

$$errorfortangent := -4.77147826613\ 10^{-7}$$

$$errorforparab := -3.10668410\ 10^{-10}$$

The errors are negative because the tangent line and the parabola are above the logarithm. The errors with the parabola are smaller than the tangent line approximation. The approximation is not very good far away from 1.

```
> plot([f,g,h], 0..3, color=[red, blue, green]);
```



We see that the parabola starts decreasing at $x = 2$, while the logarithm keeps increasing. So there is no hope further away from 2. If one wants a better approximation, one has to use higher degree polynomials. We would like to find a 3rd degree polynomial $P_3(x)$,

such that: $P_3(1) = f(1)$, P_3'(1)=f'(1),
P_3''(1)=f''(1) and P_3'''(1)=f'''(1).

Luckily Maple can gives us this polynomial, called the third degree Taylor polynomial for $\ln(x)$ at $x = 1$. The command is:

```
> x:='x'; taylor( f(x), x=1, 4 );
```

$$x := x$$

$$x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + O((x-1)^4)$$

We can plot $\ln(x)$ and the polynomial $P_3(x)$. However, first we must define a function out of it. We have the following convenient commands:

```
> taylor3:=taylor(f(x), x=1, 4);
```

$$taylor3 := x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + O((x-1)^4)$$

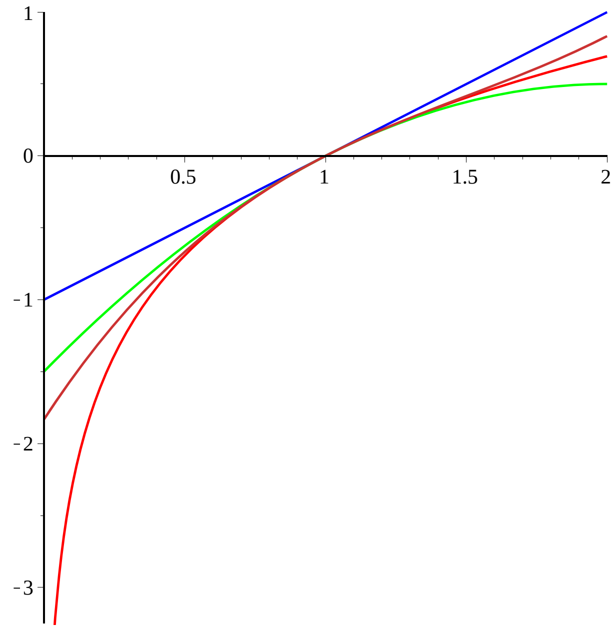
```
> lntaylor3:=convert(taylor3, polynom);
```

$$lntaylor3 := x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

```
> tay3:=unapply(lntaylor3, x);
```

$$tay3 := x \rightarrow x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

```
> plot([ln, g, h, tay3], 0..2, color=[red, blue, green, orange]);
```



We see that $P_3(x)$ (brown, orange curve) is closer to the logarithm (red curve) than the tangent line (blue) and the second degree polynomial h(x) (green).

Find the 4th, 5th, 6th degree Taylor polynomials of $\ln(x)$ **at x=1. Plot them and make a table of values for** $x = 1.1$ **and another for** $x = 1.2$ **and another for** $x = 0.9$**.**

```
> taylor( f(x), x=1, 5 ); taylor( f(x), x=1, 6 ); taylor( f(x), x=
  1, 7 );
```

$$x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+O\left((x-1)^5\right)$$

$$x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\frac{1}{5}(x-1)^5+O\left((x-1)^6\right)$$

$$x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\frac{1}{5}(x-1)^5-\frac{1}{6}(x-1)^6+O\big((x$$

$- 1)^7)$

```
> tay4:=x->x-1-(1/2)*(x-1)^2+(1/3)*(x-1)^3-(1/4)*(x-1)^4;
```

$$tay4 := x \rightarrow x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

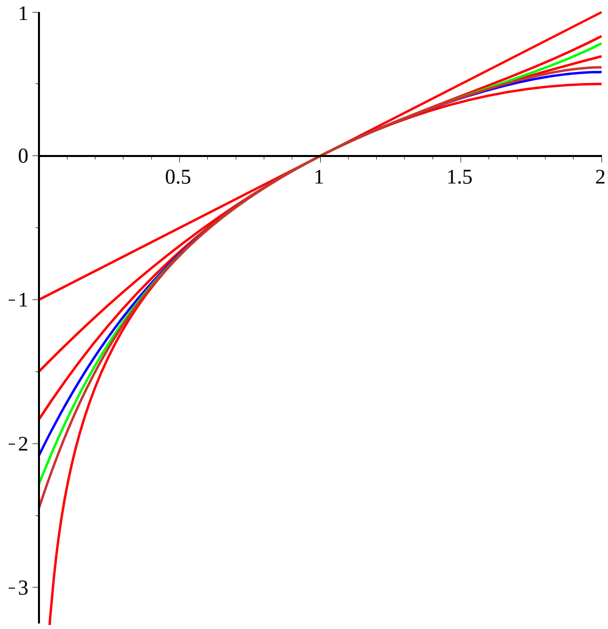
> **tay5** $:= x \rightarrow x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5;$

$$tay5 := x \rightarrow x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 \quad \textbf{(1)}$$

> **tay6** $:= x \rightarrow x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6;$

$$tay6 := x \rightarrow x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 \quad \textbf{(2)}$$

```
> plot([ln, g, h, tay3,tay4,tay5,tay6], 0..2, color=
  [red,red,red,red, blue, green, orange]);
```




```
> tay3(0.9); tay4(0.9); tay5(0.9); tay6(0.9);
```

-0.105333333333333

-0.105358333333333

-0.105360333333333

-0.105360500000000

> $tay3(1.1); tay4(1.1); tay5(1.1); tay6(1.1);$

0.0953333333333333

0.0953083333333333

0.0953103333333333

0.0953101666666666 **(3)**

> $tay3(1.2); tay4(1.2); tay5(1.2); tay6(1.2);$

0.182666666666667

0.182266666666667

0.182330666666667

(4)

0.182320000000000 **(4)**

>