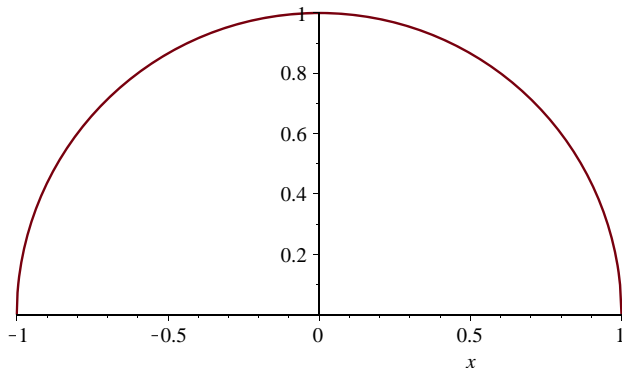# MATH 156 LAB 6

BYUNG DO PARK

Topic 1: Calculating Pi.

We have seen that $\text{Pi} = 2\int_{-1}^{1} \sqrt{1 - x^2}\, dx$. This is because the integral represents the area of a semicircle of radius 1. The circle has area Pi. In this lab you will compute the first 6 digits of pi using this integral.

Graph the fucntion $f(x) = \sqrt{1 - x^2}$ **on the range** $[-1, 1]$.

```
> f:=x->sqrt(1-x^2); plot(f(x),x=-1..1, scaling=constrained);
```

$$f := x \rightarrow \sqrt{1 - x^2}$$

Notice that we have to use scaling=constrained, which has the effect that the *x*-axis and the *y*-axis have the same scale.
As you see the function $f(x)$ is increasing on $[-1, 0]$ and decreasing on $[0, 1]$.
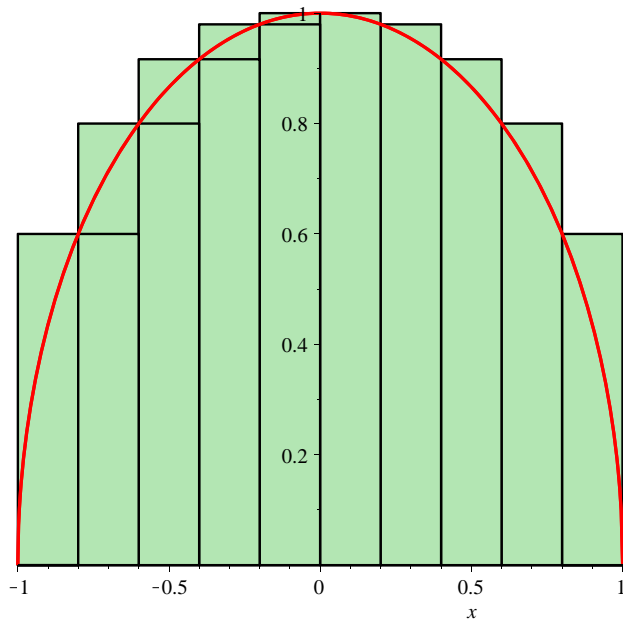
Compute the left-hand sums and right-hand sums with 10, 100, 1000 subintervals. Do not forget the student package.

```
> with(student): for k from 1 to 3 do N:=10^k; evalf(leftsum(f(x),
  x=-1..1,N)); evalf(rightsum(f(x),x=-1..1,N)); od;
```

$$N := 10$$
$$1.518524414$$
$$1.518524414$$
$$N := 100$$
$$1.569134256$$
$$1.569134256$$
$$N := 1000$$
$$1.570743739$$
$$1.570743739$$

What do you notice? Can you explain what you noticed? You may want to graph some left-hand sums and right-hand sums to explain your answer.

Left and right sums are exactly matching. This is because the function is even.

```
> lhs10:=leftbox(f(x),x=-1..1,10): rhs10:=rightbox(f(x),x=-1..1,10)
  :
```

```
> with(plots): display(lhs10,rhs10);
```

As a result we do not get underestimates and overestimates for Pi using the previous commands.

Find underestimates and overestimates for Pi**, using left-hand sums and right-hand sums with 5, 50, 500, 5000 on the intervals [-1,0] and [0,1].**

```
> for k from 1 to 4 do N:=5*10^(k-1); evalf(leftsum(4*f(x),x=-1..0,
  N)); evalf(rightsum(4*f(x),x=-1..0,N)); od;
```

$$N := 5$$

$$2.637048828$$

$$3.437048828$$

$$N := 50$$

$$3.098268512$$

$$3.178268512$$

$$N := 500$$

$$3.137487478$$
$$3.145487478$$
$$N := 5000$$
$$3.141189328$$
$$3.141989328$$

> **for** k **from** 1 **to** 4 **do** $N := 5 * 10^\wedge(k-1); evalf(leftsum(4 \cdot f(x), x = 0 ..1, N));$
  $evalf(rightsum(4 \cdot f(x), x = 0 ..1, N));$ **od**;

$$N := 5$$
$$3.437048828$$
$$2.637048828$$
$$N := 50$$
$$3.178268512$$
$$3.098268512$$
$$N := 500$$
$$3.145487476$$
$$3.137487476$$
$$N := 5000$$
$$3.141989328$$
$$3.141189328$$

(1)

We can actually predict beforehand how large n should be so that our underestimates and overestimates are within, say, epsilon. Recall that on the interval $[a, b]$ the difference $|\text{LHS}(n)\text{-RHS}(n)|=|f(b)\text{-}f(a)|\text{Delta }x$ for a monotone function. If we split $[-1,0]$ into n subintervals the error is less than $\dfrac{f(0) - f(-1)}{n} = \dfrac{1}{n}$, since Delta$x$=(b-a)/$n$. On the interval $[0,1]$, if we split into n subintervals, we have an error less than $\dfrac{f(0) - f(1)}{n} = \dfrac{1}{n}$. So in the whole interval $[-1,1]$ the error is less than $\dfrac{2}{n}$ and, since Pi is twice the integral, the error in

4

estimating Pi is less than $\dfrac{4}{n}$. If we want the error to be, say less than

0.001, we need to make $\dfrac{4}{n} < 0.001$, which gives $\dfrac{4}{0.001} < n$, i.e.,

n>4000. This explains why with 5000 subintervals we had gotten the first 3 decimals correct.

```
> 4/0.001;
```
$$4000.000000$$

How large do you need to take _n_, so that you can compute Pi with an error less than 0.0000001?

```
> 4/0.0000001;
```
$$4.000000000 \ 10^7$$

Unfortunately, this number of subintervals is too large to work out with Maple. So we resort to the trapezoid rule and the midpoint rule. **Compute TRAP(_n_), MID(_n_), with _n_=10, 100, 1000 on the whole interval [-1,1]. Which ones give underestimates and which ones give overestimates of** Pi**?** **Trap(n) gives an underestimate, and Mid(n) gives an overestimate, since the function is concave down.** **Why? What is the relation of TRAP(_n_) with LHS(_n_)?**

```
> for k from 1 to 3 do N:=10^k; evalf(0.5*(rightsum(f(x),x=-1..1,N)
  +leftsum(f(x),x=-1..1,N))); evalf(middlesum(f(x),x=-1..1,N)); od;
```
$$N := 10$$
$$1.518524414$$
$$1.585993912$$
$$N := 100$$
$$1.569134256$$
$$1.571282776$$
$$N := 1000$$
$$1.570743739$$
$$1.570811728$$

To compute more accurately, we can use Simpson's rule: SIM(_n_)=

$\dfrac{TRAP(n) + 2\,MID(n)}{3}$. There is a Maple command for

Simpson's rule:

simpson(f(x), x=lowerlimit..upperlimit, 2*number of subintervals)

Compute Pi **using Simpson's rule with** n**=10, 100, 1000,10000 subintervals. Verify your answers using the formula for Simpson's rule.**

```
> for k from 1 to 4 do N:=10^k; evalf(simpson(2*f(x),x=-1..1,2*N));
  od;
```
$$N := 10$$
$$3.127008158$$
$$N := 100$$
$$3.141133205$$
$$N := 1000$$
$$3.141578129$$
$$N := 10000$$
$$3.141592194$$

```
> N:=4; a:=-1; b:=1; delta_x:=(b-a)/N; X_Value:=[seq(a+delta_x*
  (i-1), i=1..N+1)];sum_end:=f(X_Value[1])+f(X_Value[N+1]);
```
$$N := 4$$
$$a := -1$$
$$b := 1$$
$$delta\_x := \frac{1}{2}$$
$$X\_Value := \left[-1, -\frac{1}{2}, 0, \frac{1}{2}, 1 \right]$$
$$sum\_end := 0$$

```
> evalf((delta_x/3)*(f(-1)+4*f(-1+1*delta_x)+2*f(-1+2*delta_x)+4*f
  (-1+3*delta_x)+f(-1+4*delta_x))); with(student): evalf(simpson(f
  (x),x=-1..1,2*2));
```
$$1.488033872$$
$$1.488033872$$

Topic 2: Substitutions with Maple.

All the integrals in this section can be evaluated by hand. Although Maple can help us verify our answers, you should still learn the substitution method. You are encouraged to evaluate the integrals by hand, as a test of your understanding.

Let us introduce the integral $\int \left(x^2+1\right)^4 x\,dx$: The command is Int(f(x), x)

```
> E1:=Int((x^2+1)^4*x, x); with(student):
```
$$E1 := \int \left(x^2+1\right)^4 x\,dx$$

If we know which variable to change, we introduce the substitution $u = g(x)$ the following way: changevar(u=g(x), E1,u). This substitutes u for $g(x)$ in the integral E1.

```
> E2:=changevar(u=x^2+1, E1, u);
```
$$E2 := \int \frac{1}{2}\, u^4 \, du$$

To compute this integral we can use the value command:

```
> E3:=value(E2);
```
$$E3 := \frac{1}{10}\, u^5$$

Notice that Maple does not include constants of integration. We always want to switch to the original variable x. This is done with the subs command.

```
> E4:=subs(u=x^2+1, E3);
```
$$E4 := \frac{1}{10}\, \left(x^2+1\right)^5$$

We do not need to do all these substitutions and changes, we can ask Maple to evaluate the integral E1 directly:

```
> value(E1);
```
$$\frac{1}{10}\, x^{10} + \frac{1}{2}\, x^8 + x^6 + x^4 + \frac{1}{2}\, x^2$$

The problem is that this answer does not seem to be the same as the one we got before. The reason is that we need to expand $\left(x^2+1\right)^5$.

```
> expand(E4, x);
```
$$\frac{1}{10}\, x^{10} + \frac{1}{2}\, x^8 + x^6 + x^4 + \frac{1}{2}\, x^2 + \frac{1}{10}$$

We still notice that there is a difference: the constant

$\dfrac{1}{10}$. This is so,

because in integration Maple does not care about constants. The most general antiderivative is any of the previous expressions +C. One problem is that Maple does not tell us which substitution to use. So the choice is ours and this is where our work lies.

Try to substitute $u = x^2 + 1$ **in the integral** $\displaystyle\int \left(x^2 + 1 \right)^4 dx$**. What do you notice?**

```
> E11:=Int((x^2+1)^4, x);
```
$$E11 := \int \left(x^2 + 1 \right)^4 dx$$

```
> E14:=subs(u=x^2+1, E11);
```
$$E14 := \int \left(x^2 + 1 \right)^4 dx$$

In fact one does not need substitution in this integral to compute it. We can expand the integrand:

```
> g:=expand((x^2+1)^4, x);
```
$$g := x^8 + 4\,x^6 + 6\,x^4 + 4\,x^2 + 1$$

```
> V2:=Int(g, x);
```
$$V2 := \int \left(x^8 + 4\,x^6 + 6\,x^4 + 4\,x^2 + 1 \right) dx$$

```
> V3:=value(V2);
```
$$V3 := \frac{1}{9}\,x^9 + \frac{4}{7}\,x^7 + \frac{6}{5}\,x^5 + \frac{4}{3}\,x^3 + x$$

```
> value(Int((x^2+1)^4, x));
```
$$\frac{1}{9}\,x^9 + \frac{4}{7}\,x^7 + \frac{6}{5}\,x^5 + \frac{4}{3}\,x^3 + x$$

The last command verified our previous answer.

Evaluate the integral $\displaystyle\int \cos^2 x \sin x\, dx$ **using a substitution**. Check your answer with the value command.

```
> E1:=Int((cos(x))^2*sin(x), x); with(student):
```
$$E1 := \int \cos(x)^2 \sin(x)\, dx$$

```
> E2:=changevar(u=cos(x), E1, u);
```

$$E2 := \int \left(-u^2 \right) \, du$$

```
> E3:=value(E2);
```

$$E3 := -\frac{1}{3} u^3$$

```
> E4:=subs(u=cos(x), E3);
```

$$E4 := -\frac{1}{3} \cos(x)^3$$

```
> value(E1);
```

$$-\frac{1}{3} \cos(x)^3$$

Evaluate the integral $\int e^{-x} \tan\left(e^{-x} \right) \, dx$ using a substitution. Check

your answer with the value command.

```
> E1:=Int(e^(-x)*tan(e^(-x)), x); with(student):
```

$$E1 := \int e^{-x} \tan\left(e^{-x} \right) \, dx$$

```
> E2:=changevar(u=e^(-x), E1, u);
```

$$E2 := \int \left(-\frac{\tan(u)}{\ln(e)} \right) \, du$$

```
> E3:=value(E2);
```

$$E3 := \frac{\ln(\cos(u))}{\ln(e)}$$

```
> E4:=subs(u=e^(-x), E3);
```

$$E4 := \frac{\ln\left(\cos\left(e^{-x} \right) \right)}{\ln(e)}$$

```
> value(E1);
```

$$\frac{\ln\left(\cos\left(e^{-x} \right) \right)}{\ln(e)}$$

```
>
```

9