MATH 156 LAB 8

Topic 1: Arclength

In the lecture we saw that the arclength of a function $y = f(x)$ between a and b is given by the formula: $\int_a^b \sqrt{1 + \left(\frac{\partial}{\partial x} f(x)\right)^2}\, dx$.

We will use this formula to approximate Pi. The equation of a circle of radius 1 is $x^2 + y^2 = 1$. We solve it for y to get $y = \sqrt{1 - x^2}$.

Explain why the integrand to compute the arclength of the circle is $\frac{1}{\sqrt{1 - x^2}}$.

This gives that the integral $\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1 - x^2}}\, dx = \frac{\text{Pi}}{4}$. So, to approximate Pi we can approximate the integral $\int_0^{\frac{\sqrt{2}}{2}} \frac{4}{\sqrt{1 - x^2}}\, dx$.

Introduce this function and integral and use any of the Riemann sums that you have learnt to approximate Pi **with 8 decimal digits. Make sure that you have upper and lower bounds (overestimates and underestimates) for** Pi **that allow you to compute these first 8 decimal digits.**

```
> f:=x->1/(sqrt(1-x^2)); with(student):
```

$$f := x \rightarrow \frac{1}{\sqrt{1 - x^2}}$$

```
> 2*evalf(leftsum(f(x),x=-sqrt(2)/2..sqrt(2)/2,100000));2*evalf
```

```
(rightsum(f(x),x=-sqrt(2)/2..sqrt(2)/2,100000));2*evalf(middlesum
(f(x),x=-sqrt(2)/2..sqrt(2)/2,100000));
```

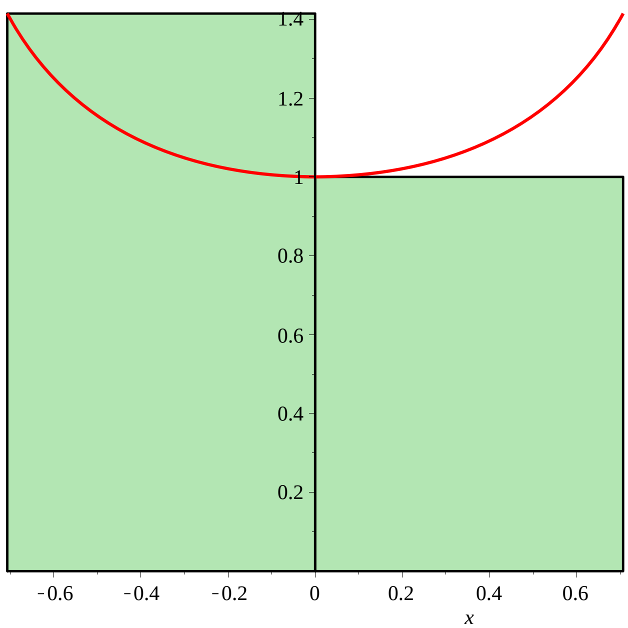
3.141592654

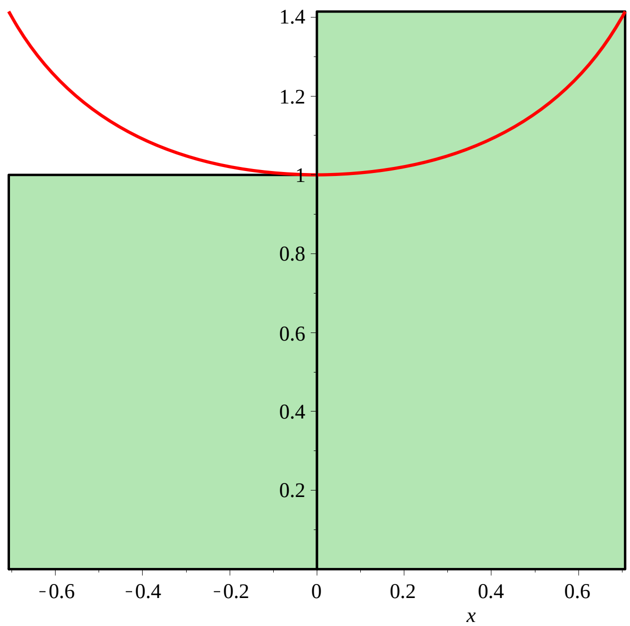
3.141592654

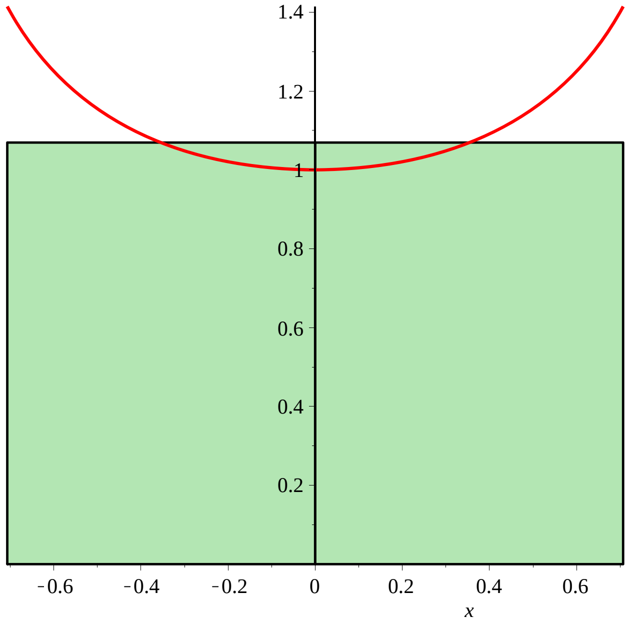
3.141592654

Explain which of these sums are overestimates, which ones are underestimates and why. Graph various Riemann sums to show your work. Graph the trapezoid sums and the midpoint sums for $n = 1, 2.$

```
> with(plots):
> leftbox(f(x),x=-sqrt(2)/2..sqrt(2)/2,2);rightbox(f(x),x=-sqrt(2)
/2..sqrt(2)/2,2);middlebox(f(x),x=-sqrt(2)/2..sqrt(2)/2,2);
```




1.4
1.2
1
0.8
0.6
0.4
0.2
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
x




```
> a:=-sqrt(2)/2;b:=sqrt(2)/2;n:=2;Dx:=(b-a)/n;
```

$$a := -\frac{1}{2}\sqrt{2}$$

$$b := \frac{1}{2}\sqrt{2}$$

$$n := 2$$

$$Dx := \frac{1}{2}\sqrt{2}$$

```
> xpoints:=[seq( a+Dx*i, i=0..n)];
```

$$xpoints := \left[-\frac{1}{2}\sqrt{2}, 0, \frac{1}{2}\sqrt{2}\right]$$

```
> valuesoflist:=map(f, xpoints);
```

$$valuesoflist := \left[\sqrt{2}, 1, \sqrt{2}\right]$$

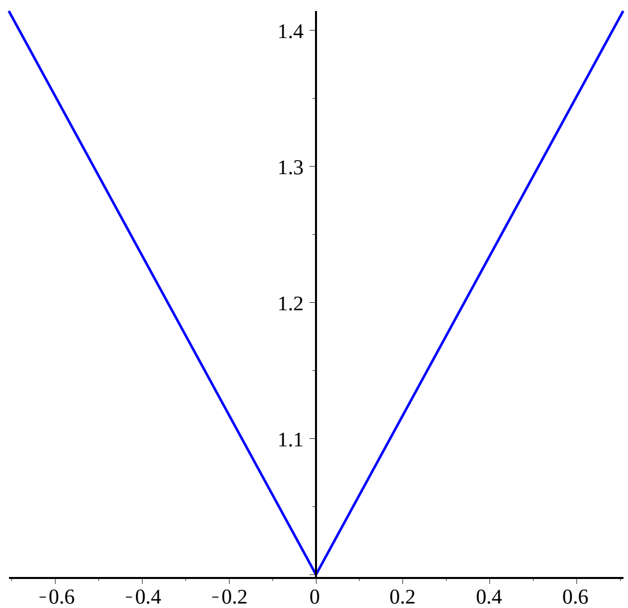
```
> pair:=(x,y)->[x,y];
```

$$pair := (x, y) \rightarrow [x, y]$$

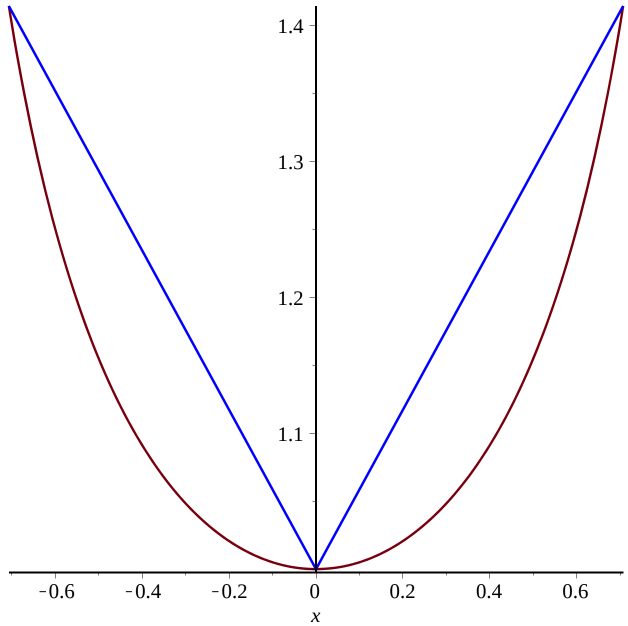
```
> datapoints:=zip(pair, xpoints,valuesoflist);
```

$$datapoints := \left[\left[-\frac{1}{2}\sqrt{2}, \sqrt{2}\right], [0, 1], \left[\frac{1}{2}\sqrt{2}, \sqrt{2}\right]\right]$$

```
> plot(datapoints, style=line, color=blue);
```



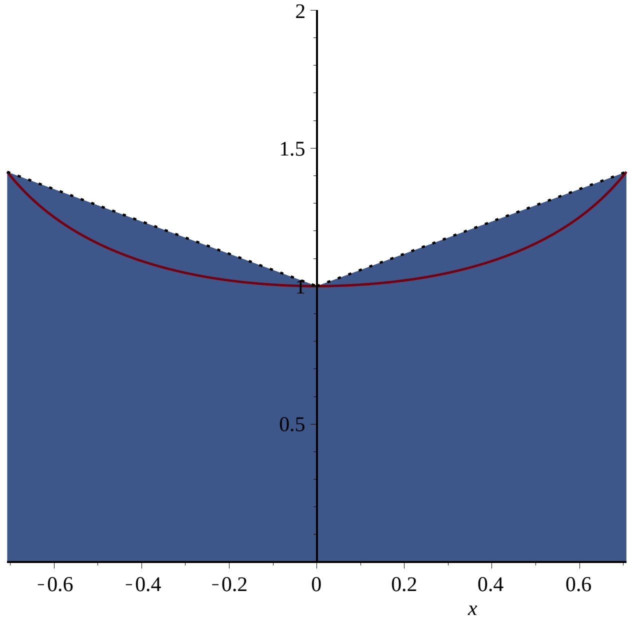

```
>  trap2:=plot(datapoints, style=line, color=blue): with(student):
  with(plots):
> display(plot(f(x), x=-sqrt(2)/2..sqrt(2)/2), trap2);
```




```
>  for i from 1 to n do trapezoid[i]:=inequal({y<f(xpoints[i])+(f
  (xpoints[i+1])-f(xpoints[i]))/Dx *(x-xpoints[i])}, x=xpoints[i]..
  xpoints[i+1], y=0..2, optionsexcluded=(color=white)): od;
> display(plot(f(x), x=-sqrt(2)/2..sqrt(2)/2),trapezoid[1],
  trapezoid[2]);
```

$$trapezoid_1 := PLOT(\ldots)$$

$$trapezoid_2 := PLOT(\ldots)$$



Now we can attack an interesting and difficult problem: the arclength of an ellipse. We will work with the ellipse $x^2 + \frac{y^2}{4} = 1$. We can solve it to get the function $y = 2\sqrt{1-x^2}$. This gives the arclength of one quarter of the ellipse to be $\int_0^1 \sqrt{\frac{1+3x^2}{1-x^2}}\, dx$. **Find the length of the whole ellipse using any method of Riemann sums that you learnt. Explain why this is the correct integral. Make sure you get underestimates and overestimates and decide how many decimal digits you have computed. Since the function is not defined at 1, we can use right-hand sum with upper limit 0.9999999.**

> **f := x→sqrt** $\left(\frac{(1+3\cdot x^2)}{(1-x^2)}\right)$; *with*(*student*) :

$$f := x \to \sqrt{\frac{1+3x^2}{1-x^2}} \quad (1)$$

> **evalf**(**rightsum**($f(x)$, $x = 0\,..0.9999999$, 100000));

$$2.460239317 \quad (2)$$

> *evalf*(*trapezoid*($f(x)$, $x = 0\,..0.9999999$, 100000));

$$2.437883641 \quad (3)$$

>

>

Let us first define the function $s(x)$ and then we can ask Maple to do the integration.

```
> s:=x->4*sqrt((1+3*x^2)/(1-x^2));
```

$$s := x \to 4\sqrt{\frac{1+3x^2}{1-x^2}}$$

```
> ellipselength:=Int(s(x), x=0..1);
```

$$ellipselength := \int_0^1 4\sqrt{\frac{3x^2+1}{-x^2+1}}\, dx$$

```
> value(ellipselength);
```

$$4\,\mathrm{EllipticE}(\mathrm{I}\sqrt{3})$$

As you see Maple cannot integrate this function. So we can only approximate the answer.

```
> t:=evalf(ellipselength);
```

$$t := 9.688448221$$

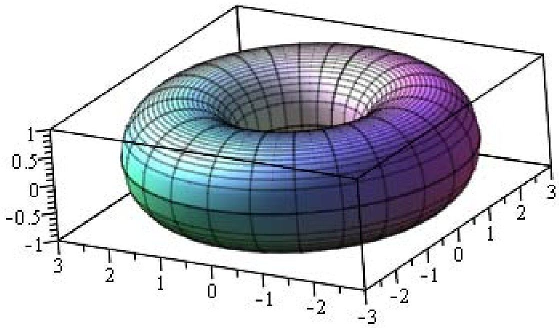
Topic 2: Some more graphing on volumes of revolution.

Humpty Dumpty decides to eat a donut, which is the solid of revolution, when we revolve around the *y*-axis the circle $(x-2)^2 + y^2 = 1$. **Plot the donut, called in mathematics torus, and compute its volume. Explain why this is the correct integral using any of the methods you learnt: discs, washers, cylindrical shells.**

```
> f:=x->sqrt(1-(x-2)^2);
```

$$f := x \to \sqrt{1-(x-2)^2}$$

```
> plot3d({[x*cos(t), x*sin(t),f(x)],[x*cos(t), x*sin(t),-f(x)]}, x=
  1..3,t=0..2*Pi, axes=BOXED, scaling=constrained);
```



```
> solve(f(y)=x,y);
```

$$2+\sqrt{-x^2+1},\, 2-\sqrt{-x^2+1}$$

```
> invf:=x->2+sqrt(-x^2+1);
```

$$invf := x \to 2+\sqrt{1-x^2}$$

```
> with(Student[Calculus1]): VolumeOfRevolution(invf(x), x=0..2,
  output=integral,axis=horizontal);
```

$$\int_0^2 \pi \left(2+\sqrt{-x^2+1}\right)^2 \, dx$$

```
> value(%);
```

$$\frac{22}{3}\pi + \pi^2 - 2\,I\,\pi \ln\left(2+\sqrt{3}\right) + 4\,I\,\pi\sqrt{3}$$

Evaluate the integral with the integration techniques you know.