MATH 156 LAB 9

Topic 1: An interesting integral.

We will consider the integral $\int \sin x \cos x \, dx$. We can substitute $u = \sin x$, which gives $du = \cos x \, dx$. Consequently the integral is $\int u \, du$. This gives $\frac{u^2}{2} = \frac{\sin^2 x}{2}$.

Introduce commands that ask Maple to compute this integral and verify the answer given.

```
> int(sin(x)*cos(x),x);
```

$$-\frac{1}{2}\cos(x)^2$$

```
> NOTE THAT THIS IS IN FACT A TRUE ANSWER.
```

Perform this substitution with Maple and show this result.

```
> E1:=Int(sin(x)*cos(x), x); with(student):
```

$$E1 := \int \sin(x)\cos(x)\,dx$$

```
> E2:=changevar(u=sin(x), E1, u);
```

$$E2 := \int u\,du$$

```
> E3:=value(E2); E4:=subs(u=sin(x), E3);
```

$$E3 := \frac{1}{2}u^2$$

$$E4 := \frac{1}{2}\sin(x)^2$$

However, we could have used the substitution $u = \cos x$, which gives $du = -\sin x \, dx$. This gives $\int (-u)\, du$. This gives $-\frac{u^2}{2} = -\frac{\cos^2 x}{2}$. As you see this answer is not the same as above. **Perform this substitution with Maple and get this result.**

```
> E11:=Int(sin(x)*cos(x), x); with(student):
```

$$E11 := \int \sin(x)\cos(x)\,dx$$

```
> E12:=changevar(u=cos(x), E11, u);
```

$$E12 := \int (-u)\, du$$

```
> E13:=value(E12); E14:=subs(u=cos(x), E13);
```

$$E13 := -\frac{1}{2} u^2$$

$$E14 := -\frac{1}{2} \cos(x)^2$$

Somehow the answers should match. It is NOT TRUE that $\sin^2 x = -\cos^2 x$. What makes the difference is that we forgot the constants of integration. So the two answers differ by a constant. To see this we ask Maple to compute the difference and simplify. **Do this.**

```
> simplify(E4-E14);
```

$$\frac{1}{2}$$

There is another way to perform the integration: We can use the trigonometric identity $\sin(2x) = 2 \sin x \cos x$. **Compute the integral using this identity and show that your answer matches with the previous two.**

```
> TrigIdent:=int((1/2)*(sin(2*x)),x);
```

$$TrigIdent := -\frac{1}{4} \cos(2x)$$

```
> TrigIdent-E4; simplify(%); TrigIdent-E14; simplify(%);
```

$$-\frac{1}{4} \cos(2x) - \frac{1}{2} \sin(x)^2$$

$$-\frac{1}{4}$$

$$-\frac{1}{4} \cos(2x) + \frac{1}{2} \cos(x)^2$$

$$\frac{1}{4}$$

Topic 2: The method of partial fractions and completing the square.

Quite often we have to integrate functions that are quotients of two polynomials

$\frac{P(x)}{Q(x)}$. These functions are called rational functions and we can use the method of partial fractions to integrate them.
Example:

$$> \int \frac{x^3 - 2x^2 + 7}{x^4 - 3x^3 + 3x^2 - 3x + 2} \, dx.$$

We introduce the expression

```
> f:=(x^3-2*x^2+7)/(x^4-3*x^3+3*x^2-3*x+2);
```

$$f := \frac{x^3 - 2x^2 + 7}{x^4 - 3x^3 + 3x^2 - 3x + 2}$$

```
> fpar:=convert(f, parfrac,x);
```

$$fpar := \frac{1}{5} \frac{13x + 6}{x^2 + 1} - \frac{3}{x - 1} + \frac{7}{5(x - 2)}$$

fpar is equal to f, only it is written in a form that is easier to integrate. Now we can define the integral of fpar and ask Maple to compute it. We recognize ourselves that $x - 2$ in the denominator will give $\ln(x - 2)$ and the $x - 1$ in the denominator will give $\ln(x - 1)$. Also we can split the expression with $x^2 + 1$ in the denominator into $\frac{6}{5(x^2 + 1)}$ plus $\frac{13x}{5(x^2 + 1)}$. The first term gives an arctan (x), while the second gives $\ln(x^2 + 1)$ with appropriate constant in front.

```
> fint:=Int(fpar, x);
```

$$fint := \int \left(\frac{1}{5} \frac{13x + 6}{x^2 + 1} - \frac{3}{x - 1} + \frac{7}{5(x - 2)} \right) dx$$

```
> fvalue:=value(fint);
```

$$fvalue := \frac{13}{10} \ln(x^2 + 1) + \frac{6}{5} \arctan(x) - 3\ln(x - 1) + \frac{7}{5} \ln(x - 2)$$

Naturally Maple can do all these things by itself:

```
> gint:=Int(f, x);
```

$$gint := \int \frac{x^3 - 2x^2 + 7}{x^4 - 3x^3 + 3x^2 - 3x + 2}\, dx$$

Find the partial fraction decomposition and integrate:

$$\int \frac{x^8 + 2x - 1}{(x-1)^3 (x^2+3)^2}\, dx .$$

```
> h:=(x^8+2*x-1)/((x-1)^3*(x^2+3)^2);
```

$$h := \frac{x^8 + 2x - 1}{(x-1)^3 (x^2+3)^2}$$

```
> hpar:=convert(h, parfrac,x);
```

$$hpar := x + 3 + \frac{1}{2(x-1)^2} + \frac{1}{32}\,\frac{-37x - 309}{x^2+3} + \frac{37}{32(x-1)} + \frac{1}{4}\,\frac{x+40}{(x^2+3)^2} + \frac{1}{8(x-1)^3}$$

```
> hint:=int(hpar,x);
```

$$hint := \frac{1}{2}x^2 + 3x - \frac{1}{2(x-1)} - \frac{37}{64}\ln(x^2+3) - \frac{767}{288}\sqrt{3}\arctan\left(\frac{1}{3}x\sqrt{3}\right) + \frac{37}{32}\ln(x-1) + \frac{1}{48}\,\frac{80x-6}{x^2+3} - \frac{1}{16(x-1)^2}$$

```
> hvalue:=value(hint);
```

$$hvalue := \frac{1}{2}x^2 + 3x - \frac{1}{2(x-1)} - \frac{37}{64}\ln(x^2+3) - \frac{767}{288}\sqrt{3}\arctan\left(\frac{1}{3}x\sqrt{3}\right) + \frac{37}{32}\ln(x-1) + \frac{1}{48}\,\frac{80x-6}{x^2+3} - \frac{1}{16(x-1)^2}$$

Sometimes this is not working because the roots of the denominator are not real numbers, or involve radicals. If the denominator does not have real roots, we can complete the square. The command is completesquare(expression, x) .

```
> f:=1/(x^2+6*x+14);
```

$$f := \frac{1}{x^2 + 6x + 14}$$

```
> fpar:=convert(f, parfrac, x);
```

$$fpar := \frac{1}{x^2 + 6x + 14}$$

```
> newf:=completesquare(f, x);
```

$$newf := \frac{1}{(x+3)^2 + 5}$$

This suggests the substitution $u = x + 3$. **Perform this substitution and compute the integral.**

```
> E1:=Int(f, x); with(student):
```

$$E1 := \int \frac{1}{x^2 + 6x + 14}\, dx$$

```
> E2:=changevar(u=x+3, E1, u);
```

$$E2 := \int \frac{1}{(u-3)^2 + 6u - 4}\, du$$

```
> E3:=value(E2);
```

$$E3 := \frac{1}{5}\sqrt{5}\arctan\left(\frac{1}{5}u\sqrt{5}\right)$$

```
> E4:=subs(u=x+3, E3);
```

$$E4 := \frac{1}{5}\sqrt{5}\arctan\left(\frac{1}{5}(x+3)\sqrt{5}\right)$$

Naturally Maple can do all these intermediate steps at once. **Write the corresponding command.**

```
> value(Int(f, x));
```

$$\frac{1}{5}\sqrt{5}\arctan\left(\frac{1}{10}(2x+6)\sqrt{5}\right)$$

Compute the integral $\displaystyle\int \frac{x^3 + 5x^2 - 7x + 1}{x^2 + x + 1}\, dx$.

```
> k:=(x^3+5*x^2-7*x+1)/(x^2+x+1);
```

$$k := \frac{x^3 + 5x^2 - 7x + 1}{x^2 + x + 1}$$

```
> kpar:=convert(k, parfrac, x);
```

$$kpar := x + 4 + \frac{-12x - 3}{x^2 + x + 1}$$

```
> kint:=int(kpar,x);
```

$$kint := \frac{1}{2}x^2 + 4x - 6\ln\left(x^2 + x + 1\right) + 2\sqrt{3}\arctan\left(\frac{1}{3}(2x+1)\sqrt{3}\right)$$

```
>
> E1:=Int(3/(x^2+x+1), x); with(student):
```

$$E1 := \int \frac{3}{x^2 + x + 1}\,dx$$

```
> E2:=changevar(u=(sqrt(3)/3)*(2*x+1), E1, u);
```

$$E2 := \int \frac{3}{2}\,\frac{\sqrt{3}}{\frac{1}{12}\left(\sqrt{3} - 3u\right)^2 - \frac{1}{6}\left(\sqrt{3} - 3u\right)\sqrt{3} + 1}\,du$$

```
> E3:=value(E2);
```

$$E3 := 2\sqrt{3}\arctan(u)$$

```
> E4:=subs(u=(sqrt(3)/3)*(2*x+1), E3);
```

$$E4 := 2\sqrt{3}\arctan\left(\frac{1}{3}(2x+1)\sqrt{3}\right)$$

```
>
>
```

Verify the answer by doing the integration by hand.