# Final Exam

Spring 2014, MAT156 Section 03LB[51316]

May 20th, 2014. 11:00AM--12:40PM.

## Your Name:

Instructions: You can use only MAPLE program and a web browser(only for the purpose of submitting your exam solution to the instructor). You may not use any other programs other than these two. You can look up your MAPLE source files, but otherwise this exam is closed-book, closed-note, and you may not use any electronic device in this exam except your PC. You are not allowed to talk to other students. Type all details explicitly. All solutions should be obtained by using MAPLE codes.

Don't forget to run student package

```
> with(student) : with(Student[Calculus1]) :
```

Problem 1. (10 points) Evaluate $\int e\^(2\,x)\,\tan(e\^(2\,x))\,dx$ using a substitution. (10 points)

```
> E1:=Int(e^(2*x)*tan(e^(2*x)), x); with(student):
```

$$\int e^{2x}\tan\left(e^{2x}\right)\,dx \quad (1)$$

```
> E2:=changevar(u=e^(2*x), E1, u);
```

$$\int \frac{1}{2}\,\frac{\tan(u)}{\ln(e)}\,du \quad (2)$$

```
> E3:=value(E2);
```

$$-\frac{1}{2}\,\frac{\ln(\cos(u))}{\ln(e)} \quad (3)$$

```
> E4:=subs(u=e^(2*x), E3);
```

$$-\frac{1}{2}\,\frac{\ln\left(\cos\left(e^{2x}\right)\right)}{\ln(e)} \quad (4)$$

```
> value(E1);
```

$$-\frac{1}{2}\,\frac{\ln\left(\cos\left(e^{2x}\right)\right)}{\ln(e)} \quad (5)$$

Problem 2. (30 points) Using $\int_a^b \frac{k}{\sqrt{1-x^2}}\,dx$, answer the following questions:

(1) Determine constants k, a, b so that the integral has its value Pi. (5 points)

(2) Give two different methods for approximating the integral so that one is underestimating and another is overestimating the integral. (10 points. Justification is required.)
(3) Using two approximation methods you have chosen in (2), give an approximate value of Pi up to 5th decimal digit. (5 points)
(4) Among simpson's rule, midpoint sum, trapezoid method, leftsum, and right sum, which one is the most efficient in approximating Pi, based on the result of (3)? (10 points)

```
> (1) a=0, b=1/sqrt(2), k=4. Then we get:
> int(4/sqrt(-x^2+1), x=0..1/sqrt(2));
```

$$\pi \tag{6}$$

```
> (2) Use trapezoid method for overestimate, and middlesum for
  underestimate. Both are because 4/sqrt(-x^2+1) is concave up.
> (3) f:=x->4/sqrt(1-x^2);
> evalf(trapezoid(f(x),x=0..1/sqrt(2), 1000));
> evalf(middlesum(f(x),x=0..1/sqrt(2), 1000));
```

$$x \to \frac{4}{\sqrt{1-x^2}} \tag{7}$$

3.141592985
3.141592486

```
> So the answer is 3.14159
> (4)
> evalf(rightsum(f(x),x=0..1/sqrt(2), 100));
```

3.147483849 **(8)**

```
> evalf(trapezoid(f(x),x=0..1/sqrt(2), 100));
```

3.141625985 **(9)**

```
> evalf(simpson(f(x),x=0..1/sqrt(2), 100));
```

3.141592659 **(10)**

```
> evalf(leftsum(f(x),x=0..1/sqrt(2), 100));
```

3.135768120 **(11)**

```
> evalf(middlesum(f(x),x=0..1/sqrt(2), 100));
```

3.141575988 **(12)**

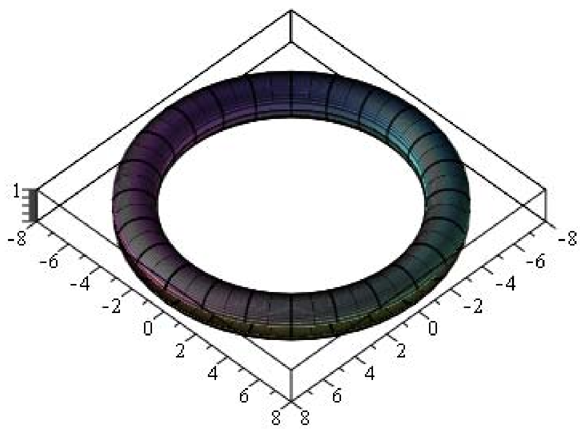
```
> Hence the Simpson's rule is supposedly the most efficient in
  approximating Pi in this choice of integral representation of Pi.
```

Problem 3. (20 points, 10 points each) A torus, which is the solid of revolution, when we revolve around the y-axis the circle
(x-7)^2+y2 = 1. (1) Plot the torus. (2) Compute its volume.

```
> with(Student[Calculus1]): f:=x->sqrt(1-(x-7)^2);
```

$$x \to \sqrt{1-(x-7)^2} \tag{13}$$

```
> plot3d({[x*cos(t), x*sin(t),f(x)],[x*cos(t), x*sin(t),-f(x)]}, x=
  6..8,t=0..2*Pi, axes=BOXED, scaling=constrained);
```




```
> invf:=solve(f(y)=x,y);
```

$$7+\sqrt{-x^2+1},\ 7-\sqrt{-x^2+1} \quad (14)$$

```
> outer:=VolumeOfRevolution(invf[1],x=-1..1, output=integral,axis=
  horizontal);
```

$$\int_{-1}^{1} \pi\left(7+\sqrt{-x^2+1}\right)^2 dx \quad (15)$$

```
> iNner:=VolumeOfRevolution(invf[2],x=-1..1, output=integral,axis=
  horizontal);
```

$$\int_{-1}^{1} \pi\left(-7+\sqrt{-x^2+1}\right)^2 dx \quad (16)$$

```
> value(outer-iNner);
```

$$14\,\pi^2 \quad (17)$$

Problem 4. (10 points) Using integration by parts, integrate exp(-2*x)*sin(3*x).

integrate int by part, plotting dotted graph for partial sum, taylor

```
> E1:=Int(exp(-2*x)*sin(3*x),x);
```

$$\int e^{-2x}\sin(3x)\,dx \tag{18}$$

```
> E2:=intparts(E1,sin(3*x));
```

$$-\frac{1}{2}e^{-2x}\sin(3x)-\left(\int\left(-\frac{3}{2}\cos(3x)\,e^{-2x}\right)dx\right) \tag{19}$$

```
> E21:=Int(-(3/2)*cos(3*x)*exp(-2*x), x);
```

$$\int\left(-\frac{3}{2}\cos(3x)\,e^{-2x}\right)dx \tag{20}$$

```
> E22:=intparts(E21,cos(3*x));
```

$$\frac{3}{4}\cos(3x)\,e^{-2x}-\left(\int\left(-\frac{9}{4}e^{-2x}\sin(3x)\right)dx\right) \tag{21}$$

```
> E2new:=(-1/2)*exp(-2*x)*sin(3*x)-E22;
```

$$-\frac{1}{2}e^{-2x}\sin(3x)-\frac{3}{4}\cos(3x)\,e^{-2x}+\int\left(-\frac{9}{4}e^{-2x}\sin(3x)\right)dx \tag{22}$$

```
> Answer:=simplify((1+9/4)^(-1)*(E2new+(9/4)*E1));
```

$$-\frac{1}{13}e^{-2x}(2\sin(3x)+3\cos(3x)) \tag{23}$$

Problem 5. (10 points, 5 points each) Let f(x)=1/(1-x).
(1) Find the Taylor polynomial of order 2,3,4,5 expanded at x=2.
(2) Plot f(x), straight line, quadratic, cubic, quartic, and cubic approximations of f(x)in a single graph, using colors red, blue, green, purple, orange, and yellow in the same order. Use x=-5..5, y=-5..5.

```
> (1) f:=x->1/(1-x);
```

$$x\rightarrow\frac{1}{1-x} \tag{24}$$

```
> tay2:=taylor(f(x),x=2,2); tay3:=taylor(f(x),x=2,3); tay4:=taylor
  (f(x),x=2,4); tay5:=taylor(f(x),x=2,5);
```

$$-1+x-2+O\left((x-2)^2\right) \tag{25}$$
$$-1+x-2-(x-2)^2+O\left((x-2)^3\right)$$
$$-1+x-2-(x-2)^2+(x-2)^3+O\left((x-2)^4\right)$$
$$-1+x-2-(x-2)^2+(x-2)^3-(x-2)^4+O\left((x-2)^5\right)$$

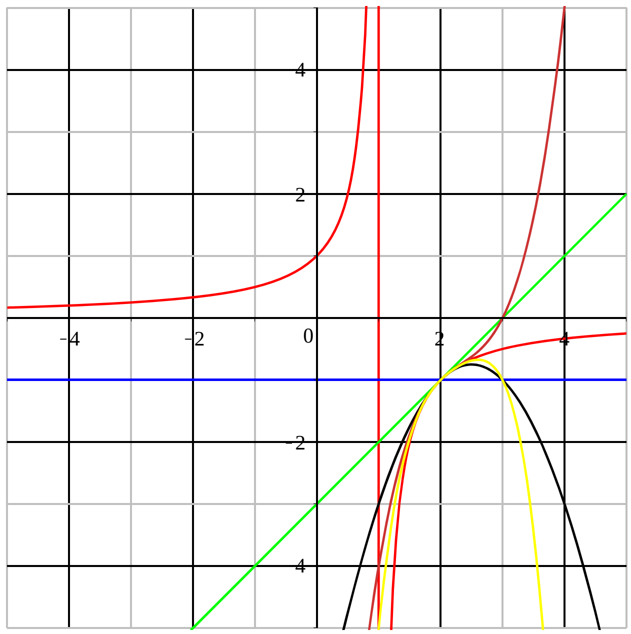
```
> (2) tay1:=taylor(f(x),x=2,1);
```

$$-1+O(x-2) \tag{26}$$

```
> tay1:=convert(tay1,polynom); tay2:=convert(tay2,polynom); tay3:=
  convert(tay3,polynom); tay4:=convert(tay4,polynom); tay5:=convert
  (tay5,polynom);
```

$$-1 \tag{27}$$
$$-3+x$$
$$-3+x-(x-2)^2$$
$$-3+x-(x-2)^2+(x-2)^3$$
$$-3+x-(x-2)^2+(x-2)^3-(x-2)^4$$

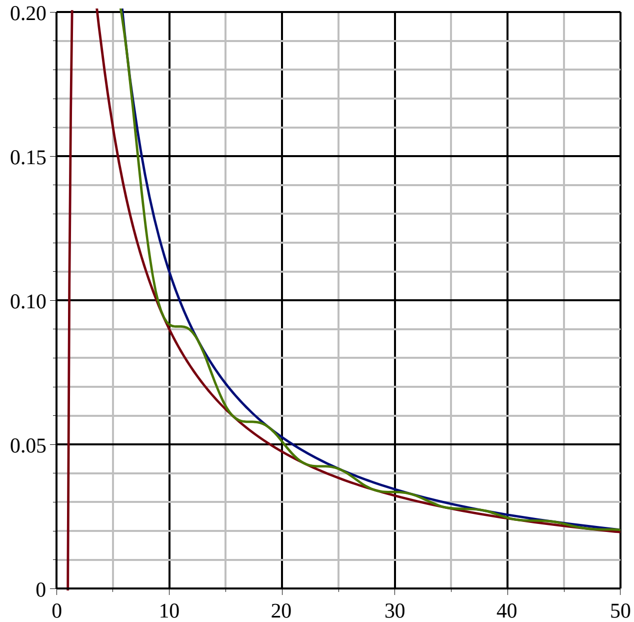
```
> plot([f(x),tay1,tay2,tay3,tay4,tay5],x=-5..5, y=-5..5, color=
  [red,blue,green,black,orange,yellow]);
```




Problem 6.(10 points) Decide whether the improper integral $\int_{2}^{\infty} \frac{\cos(x) + x}{x^2} \, dx$ converges or not.

Hint: Draw the integrand function and (1+x)/x^2, (-1+x)/x^2 on the same graph. By doing this, you can guess an inequality. Use if f(x)‰44g(x), then $\int_{a}^{b} f(x) \, dx \le \int_{a}^{b} g(x) \, dx$.

```
> plot({(cos(x)+x)/x^2,(1+x)/x^2,(-1+x)/x^2},x=0..50, y=0..0.2);
```




```
> In fact (cos(x)+x)/x^2 is greater than equal to (-1+x)/x^2, and
  is everywhere positive from 2 to infinity. Since
> Int((-1+x)/x^2,x=2..infinity);
```

$$\int_{2}^{\infty} \frac{-1+x}{x^2}\, dx \tag{28}$$

```
> value(%);
```

$$\infty \tag{29}$$

```
> we conclude that the given integral is divergent.
```

Problem 7. (10 points) Calculate $\sum_{n=0}^{\infty} 50 \left(\frac{4}{7}\right)^n$ using MAPLE.

```
> partialsum:=N->sum(50*(4/7)^n, n=0..N);
```

$$N \rightarrow \sum_{n=0}^{N} 50 \left(\frac{4}{7}\right)^n \tag{30}$$

```
> limit(partialsum(N),N=infinity);
```

(31)

$$\frac{350}{3} \tag{31}$$