We want to show that

$$\lim_{(a,b)\to(0,0)} \frac{|a||b|}{\sqrt{a^2+b^2}} = 0.$$

Proof. Notice that $|a| \leq \sqrt{a^2+b^2}$ and $|b| \leq \sqrt{a^2+b^2}$ and also that

$$\frac{|a||b|}{\sqrt{a^2+b^2}} \leq \frac{a^2+b^2}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2}.$$

Now for every $\varepsilon > 0$ there is $\delta = \varepsilon$ such that $0 < \|(a,b)-(0,0)\| = \sqrt{a^2+b^2} < \delta$ implies

$$\left|\frac{|a||b|}{\sqrt{a^2+b^2}} - 0\right| \leq \sqrt{a^2+b^2} < \delta = \varepsilon.$$

□

Another solution suggested by Mr. Simon Weil.

Proof. Pick any path $\alpha : [0,1] \to \mathbb{R}^2$ such that $\alpha(0) = (a,b)$ and $\alpha(1) = (0,0)$. Let $r = |\alpha(t)|$ and θ be the angle between the vector $\alpha(t)$ and the x-axis measured counterclockwise.

$$\begin{aligned}\lim_{(a,b)\to(0,0)} \frac{|a||b|}{\sqrt{a^2+b^2}} &= \lim_{t\to1} \frac{r|\cos(\theta)|\cdot r|\sin(\theta)|}{\sqrt{|r^2\cos^2\theta + r^2\sin^2\theta}} \\ &= \lim_{t\to1} \frac{r^2|\frac{1}{2}\sin(2\theta)|}{|r|} = \lim_{t\to1} r|\frac{1}{2}\sin(2\theta)| = 0.\end{aligned}$$

To see the last equality is indeed the case, one can use the squeeze theorem. Also notice that r and θ both are functions in t where $r \to 0$ as $t \to 1$. □