Consider a function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function is differentiable every $x \in \mathbb{R}$ but continuous at all $x \in \mathbb{R}$ except $x = 0$.

Proof. **Step I:** Let $x \neq 0$. Then

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

This shows that the function f is differentiable at x and f' is continuous at x as long as $x \neq 0$.

Step II: We claim that f is differentiable at $x = 0$. By definition,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h}.$$

An application of the squeeze theorem using

$$-h \leq \frac{h^2 \sin \frac{1}{h}}{h} = h \sin \frac{1}{h} \leq h$$

as h tends to 0 implies that $f'(0) = 0$. Notice that $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left[2x \sin \frac{1}{x} - \cos \frac{1}{x}\right]$ is not 0 because $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist. □