Exercise 3.3.31. Write the number 120 as a sum of three numbers so that the sum of the products taken two at a time is a maximum.

Solution. Let $x + y + z = 120$ or $z = 120 - x - y$. We want to find a global maximum of $f(x, y) = xy + y(120 - x - y) + x(120 - x - y)$ on $\mathbb{R}^2$ if exists. Since f is clearly a C^1-function, if there is any global maximum it must be a critical point while being a local maximum; i.e. $\nabla f(x_0, y_0) = 0$. By rewriting this equation, we get $120 - 2x_0 - y_0 = 0$ and $120 - 2y_0 - x_0 = 0$. By solving this system of equations we get $(x_0, y_0) = (40, 40)$.

We verify that, at (x_0, y_0) we have a local maximum. Since $f_{xx} = -2$, $f_{yy} = -2$, and $f_{xy} = -1$ for all $(x, y) \in \mathbb{R}^2$, we see that the Hessian of f at (x_0, y_0) is also

$$Hf(x_0, y_0)(\overrightarrow{h}) = (\overrightarrow{h})^T \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \overrightarrow{h}.$$

Since the $(1, 1)$ entry is negative and $|Hf(x_0, y_0)| = 3 > 0$, by the second derivative test, the function f of class C^2 attains strict local maximum at (x_0, y_0).

We claim that this local maximum is the global maximum, because, for any (x, y), the $(1, 1)$ entry of the Hessian of f is negative and $|Hf(x, y)| = 3 > 0$, and hence there is no other local maxima or saddle points which might result in bigger values of f. Therefore we conclude that when three numbers are 40, 40, and 40, the product attains the maximum. □