Take-home Midterm Exam (Corrected version)

MATH 250 Section 02

From April 13th, 2016 7:25pm to April 20th, 2016 5:35pm.

<u>Deadline:</u> **April 20th, 2016 5:35pm.**

Instructions: Policies of this exam are described on coversheet. You can keep this problem sheet. Please submit your solutions with the coversheet stapled on the top. Please note that <u>late-submissions are NOT accepted</u>.

1. Prove the following result: $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c).$

2. Prove if the following statement is true, or disprove by giving an example if it is false:
Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function on A whose all first partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ exist at $\overrightarrow{x}_0 \in A$. Then the function f is continuous at $\overrightarrow{x}_0 \in A$.

3. Determine whether the following functions is differentiable:

$$f(x,y) = \frac{x}{y} + \frac{y}{x} \text{ if } x \text{ and } y \text{ both are nonzero and } f(x,y) = 0 \text{ if } x = 0 \text{ or } y = 0.$$

4. Find a unit vector normal to the surface S given by $x^3y^3 + y - z = 1$ at $\overrightarrow{x}_0 = (1,1,1)$.

5. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable at $\overrightarrow{x}_0 \in \mathbb{R}^3$. Prove that

$$\lim_{\overrightarrow{x} \to \overrightarrow{x}_0} \frac{|f(\overrightarrow{x}) - f(\overrightarrow{x}_0)|}{\|\overrightarrow{x} - \overrightarrow{x}_0\|}$$

is bounded by a positive constant. (Hint: Use the triangle inequality and the Cauchy-Schwarz inequality)

Please see overleaf

6. Let j be the coordinate change map from the spherical coordinate to the cartesian coordinate defined by

$$x = r\cos\theta\sin\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\phi$$

Also let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable map. Calculate $D(f \circ j)$.

7. Let $f(x, y) = x^5 + y^4 + 3x^2 + 2xy + 2x + y^2 + 2y + 1$. Find the second-order Taylor approximation of f at $(1, 0)$.

8. For given $f(x, y, z) = x^2 + y^2 + z^2 - xyz$ find all critical points and determine whether they are local minima, local maxima, saddle points, or none of them.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 - y^2$, and S the unit circle in $\mathbb{R}^2$. Find the extrema of $f|_S$ by using the bordered Hessian test. (No credit will be given if there is no use of bordered Hessian test.)

10. Let $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Find the absolute maximum and minimum values of f on the elliptical region $x^2 + \frac{1}{2}y^2 \leq 1$.