Solutions to take-home Midterm Exam
MATH 250 Section 02
From April 13th, 2016 7:25pm to April 20th, 2016 5:35pm.

1. Prove the following result: $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c).$

Solution. Expand the given determinant along the first column. We get

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \begin{vmatrix} b & b^3 \\ c & c^3 \end{vmatrix} - \begin{vmatrix} a & a^3 \\ c & c^3 \end{vmatrix} + \begin{vmatrix} a & a^3 \\ b & b^3 \end{vmatrix} = bc^3 - cb^3 - (ac^3 - ca^3) + ab^3 - ba^3.$$

By rearranging terms, we see the far RHS is equal to the following. Factor out $(b-c)$.

$$\begin{aligned} &bc^3 - cb^3 + ab^3 - ac^3 - (ba^3 - ca^3) = -bc(b^2 - c^2) + a(b^2 + bc + c^2)(b-c) - a^3(b-c) \\ =& -(b-c)\big(a^3 - (b^2 + bc + c^2)a + bc(b+c)\big) = -(b-c)(a-b)(a-c)(a+b+c) \\ =&(a-b)(b-c)(c-a)(a+b+c). \end{aligned}$$

□

+5 for knowing determinant expansion.

2. Prove if the following statement is true, or disprove by giving an example if it is false:
Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function on A whose all first partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ exist at $\vec{x}_0 \in A$. Then the function f is continuous at $\vec{x}_0 \in A$.

Solution. False. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0 \\ 0 & \text{if } x \neq 0 \text{ and } y \neq 0 \end{cases}$$

This function has $f_x(0,0)$ and $f_y(0,0)$;

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

However the function is not continuous at the origin. In the case $(x,y) \to (0,0)$ along x- or y-axes, $\lim_{(x,y)\to(0,0)} f(x,y) = 1 = f(0,0)$ whereas $(x,y) \to (0,0)$ along the diagonal path $\{(x,y) \in \mathbb{R}^2 : x = y\}$, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ whereas $f(0,0) = 1$. □

+5 for giving a right example. +3 for verifying existence of partial derivatives, +2 for verifying discontinuity at $\vec{x_0} \in A$.

3. Determine whether the following functions is differentiable:

$$f(x,y) = \frac{x}{y} + \frac{y}{x} \text{ if } x \text{ and } y \text{ both are nonzero and } f(x,y) = 0 \text{ if } x = 0 \text{ or } y = 0.$$

Solution. This question is asking if the given function is differentiable at every $(x_0, y_0) \in \mathbb{R}^2$.
Recall that

Proposition 1. If f is of class C^1 at (x_0, y_0) (i.e., f has all first partial derivatives at (x_0, y_0) and the partial derivatives are continuous at at (x_0, y_0)), then f is differentiable.

We claim that our f is of class C^1 at (x_0, y_0) with nonzero x_0 and y_0, since it has partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{y} - \frac{y}{x^2} \\ \frac{\partial f}{\partial y} &= -\frac{x}{y^2} + \frac{1}{x} \end{aligned} \tag{1}$$

and these are continuous functions at (x_0, y_0) if x_0 and y_0 are nonzero. Therefore, by the Proposition 1, differentiability of f follows.

Now let $x_0 \neq 0$ and $y_0 = 0$. We first calculate partial derivatives of f at $(x_0, 0)$ if exists.

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(x_0,0)} &= \lim_{h\to 0} \frac{f(x_0+h,0) - f(x_0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \\ \left.\frac{\partial f}{\partial y}\right|_{(x_0,0)} &= \lim_{h\to 0} \frac{f(x_0,0+h) - f(x_0,0)}{h} = \lim_{h\to 0} \frac{\frac{x_0}{h} + \frac{h}{x_0}}{h} = \lim_{h\to 0} \frac{x_0}{h^2} + \frac{1}{x_0} : \text{Does not exist.} \end{aligned} \tag{2}$$

By similar calculations we conclude that $\left.\frac{\partial f}{\partial x}\right|_{(0,y_0)}$ does not exist and $\left.\frac{\partial f}{\partial y}\right|_{(0,y_0)} = 0$ if $x_0 = 0$ and $y_0 \neq 0$. In these two cases, since not all partial derivatives exist, f is not differentiable.

Consider the case that $x_0 = y_0 = 0$. We have $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 = \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$. Now we check whether f has a good approximation at $(0,0)$. In other words, we check if the following equality holds:

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f(h_1,h_2) - f(0,0) - \nabla f(0,0)\cdot(h_1,h_2)|}{\|(h_1,h_2)\|} = 0.$$

We see that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f((h_1,h_2)) - 0 - 0|}{\|(h_1,h_2)\|} = \lim_{(h_1,h_2)\to(0,0)} \frac{\left|\frac{h_1}{h_2} + \frac{h_2}{h_1}\right|}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{\left|\frac{h_1^2+h_2^2}{h_1 h_2}\right|}{\sqrt{h_1^2+h_2^2}}.$$

We claim that the far RHS is not zero.

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\left|\frac{h_1^2+h_2^2}{h_1 h_2}\right|}{\sqrt{h_1^2+h_2^2}} = \lim_{r\to 0} \frac{\left|\frac{r^2}{r^2\sin\theta\cos\theta}\right|}{r} = \lim_{r\to 0} \left|\frac{1}{r\sin\theta\cos\theta}\right|$$

which is not necessarily vanishing. Notice that we have used a substitution $h_1 = r\cos\theta$ and $h_2 = r\sin\theta$ in the first equality. Therefore, we conclude that f is not differentiable at (x_0, y_0) if one of coordinates is zero. $\square$

Any correct proof that f is not differentiable everywhere on $\mathbb{R}^2$ gets credit.

4. Find a unit vector normal to the surface S given by $x^3y^3 + y - z = 1$ at $\overrightarrow{x}_0 = (1, 1, 1)$.

Solution. The given surfact S is the level surface of $g(x, y, z) = 1$ for $g(x, y, z) := x^3y^3 + y - z$. The gradient vector at $\overrightarrow{x}_0$ is a vector that is normal to S, whereas $\nabla g(\overrightarrow{x}_0) = (3x^2y^3, 3x^3y^2+1, -1)|_{\overrightarrow{x}_0} = (3, 4, -1)$. By normalizing, we get

$$\frac{\nabla g(\overrightarrow{x}_0)}{\|\nabla g(\overrightarrow{x}_0)\|} = \left(\frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{-1}{\sqrt{26}}\right).$$

□

-2 for not normalizing

5. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable at $\overrightarrow{x}_0 \in \mathbb{R}^3$. Prove that

$$\lim_{\overrightarrow{x} \to \overrightarrow{x}_0} \frac{|f(\overrightarrow{x}) - f(\overrightarrow{x}_0)|}{\|\overrightarrow{x} - \overrightarrow{x}_0\|}$$

is bounded by a positive constant. (Hint: Use the triangle inequality and the Cauchy-Schwarz inequality)

Solution. Recall the following inequalities:

$$\|\overrightarrow{v}\| + \|\overrightarrow{w}\| \leq \|\overrightarrow{v} + \overrightarrow{w}\| \qquad \text{for any } \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n. \tag{3}$$

called the triangle inequality, and

$$|\overrightarrow{v} \cdot \overrightarrow{w}| \leq \|\overrightarrow{v}\|\|\overrightarrow{w}\| \qquad \text{for any } \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n. \tag{4}$$

which is called the Cauchy-Schwarz inequality. Note that the following inequality is nothing but a restatement of (3).

$$\|\overrightarrow{v}\| - \|\overrightarrow{w}\| \leq \|\overrightarrow{v} - \overrightarrow{w}\| \qquad \text{for any } \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n. \tag{5}$$

This is because $\|\overrightarrow{v}\| = \|\overrightarrow{v} - \overrightarrow{w} + \overrightarrow{w}\|$.

Now we prove the given statement using the above inequalities. Recall that f is differentiable at $\overrightarrow{x}_0$ if partial derivatives $f_{x_1}(\overrightarrow{x}_0), \cdots, f_{x_n}(\overrightarrow{x}_0)$ exists and

$$\lim_{\overrightarrow{h} \to \overrightarrow{0}} \frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0) - \nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} = 0.$$

Recall the $\epsilon - \delta$ definition of limit. (Section 2.2.) Having the above limit *implies* that there exists some $\delta > 0$ such that $0 < \|\overrightarrow{h}\| < \delta$ implies $\left|\frac{|f((\overrightarrow{x}_0+\overrightarrow{h})-f(\overrightarrow{x}_0)-\nabla f(\overrightarrow{x}_0)\cdot\overrightarrow{h}|}{\|\overrightarrow{h}\|} - 0\right| < 1$. (Note: We chose $\varepsilon = 1$ which we can.)

Therefore we look at

$$\frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0) - \nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} < 1.$$

By (5), we observe that

$$\frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0)| - |\nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} \leq \frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0) - \nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} < 1$$

Hence

$$\frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0)|}{\|\overrightarrow{h}\|} < 1 + \frac{|\nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|},$$

and notice that by (4),

$$1 + \frac{|\nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} \leq 1 + \frac{\|\nabla f(\overrightarrow{x}_0)\|\|\overrightarrow{h}\|}{\|\overrightarrow{h}\|} = 1 + \|\nabla f(\overrightarrow{x}_0)\|$$

where the far RHS is a constant that is positive. □

6. Let j be the coordinate change map from the spherical coordinate to the cartesian coordinate defined by

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{aligned}$$

Also let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable map. Calculate $D(f \circ j)$.

Solution.

$$\begin{aligned} j : \mathbb{R}^+ \times (0, 2\pi) \times (0, \pi) &\to \mathbb{R}^3 \\ (r, \theta, \phi) &\mapsto (x, y, z) \end{aligned}$$

and

$$D(j)(r,\theta,\phi) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}(r,\theta,\phi) = \begin{pmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix}$$

By the chain rule,

$$D(f \circ j)(r,\theta,\phi) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}(x,y,z) \begin{pmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix}(r,\theta,\phi)$$

Therefore

$$\frac{\partial(f\circ j)}{\partial r} = \cos\theta\sin\phi\frac{\partial f}{\partial x} + \sin\theta\sin\phi\frac{\partial f}{\partial y} + \cos\phi\frac{\partial f}{\partial z}$$
$$\frac{\partial(f\circ j)}{\partial \theta} = -r\sin\theta\sin\phi\frac{\partial f}{\partial x} + r\cos\theta\sin\phi\frac{\partial f}{\partial y}$$
$$\frac{\partial(f\circ j)}{\partial \phi} = r\cos\theta\cos\phi\frac{\partial f}{\partial x} + r\sin\theta\cos\phi\frac{\partial f}{\partial y} - r\sin\phi\frac{\partial f}{\partial z}$$

□

+5 for correct calculation of $D(j)$.

7. Let $f(x,y) = x^5 + y^4 + 3x^2 + 2xy + 2x + y^2 + 2y + 1$. Find the second order Taylor approximation of f at $(1,0)$.

Solution. Let $\vec{x} = (x,y)$ and $\vec{x}_0 = (1,0)$. Recall that

$$f(\vec{x}-\vec{x}_0) = f(\vec{x}_0) + \nabla f(\vec{x}_0)\cdot(\vec{x}-\vec{x}_0) + \frac{1}{2!}(\vec{x}-\vec{x}_0)^T Hf(\vec{x}_0)(\vec{x}-\vec{x}_0) + R_2(\vec{x}_0, \vec{x}-\vec{x}_0).$$

Here $(\vec{x}-\vec{x}_0)^T$ denotes the transpose of the column vector $\vec{x}-\vec{x}_0$.

Since $f_x = 5x^4+6x+2y+2$, $f_y = 4y^3+2x+2y+2$, $f_{xx} = 20x^3+6$, $f_{xy} = 2$, and $f_{yy} = 12y^2+2$, we have

$$\begin{aligned} f(\vec{x}-\vec{x}_0) &= 7 + (13,4)\cdot(x-1,y) + \frac{1}{2!}(x-1,y)^T\begin{pmatrix} 26 & 2 \\ 2 & 2\end{pmatrix}(x-1,y) + R_2((1,0),(x-1,y)) \\ &= 7 + 13(x-1) + 4y + 13(x-1)^2 + (x-1)y + (x-1)y + y^2 + R_2((1,0),(x-1,y)) \end{aligned}$$

Therefore the second order Taylor approximation $P_2(x,y)$ of f is

$$P_2(x,y) = 7 + 13(x-1) + 4y + 13(x-1)^2 + 2(x-1)y + xy + y^2.$$

□

+5 for knowing Taylor approximation. Each incorrect term gets -1 for minor errors, -3 if the order of the term is not correct.

8. For given $f(x,y,z) = x^2+y^2+z^2-xyz$ find all critical points and determine whether they are local minima, local maxima, saddle points, or none of them.

Solution. **Step 1:** We find all critical points.

$$f_x = 2x - yz = 0$$
$$f_y = 2y - xz = 0$$
$$f_z = 2z - xy = 0$$

By solving this system of equations, we get $xyz = 0$ or $xyz = 8$. In the case of the former, by using each of the equations, we conclude $x = y = z = 0$, whereas in the case of the latter, we obtain $(x, y, z) = (2, 2, 2)$, $(-2, -2, 2)$, $(-2, 2, -2)$, and $(2, -2, -2)$.

Step 2: We compute the Hessian matrix Hf at each of the above points. Observe that

$$\begin{aligned} f_{xx} &= +2 & f_{yx} &= -z & f_{zx} &= -y \\ f_{xy} &= -z & f_{yy} &= +2 & f_{zy} &= -x \\ f_{xz} &= -y & f_{yz} &= -x & f_{zz} &= +2 \end{aligned}$$

So we get

$$Hf(0,0,0) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}, \quad H_1 = 2, \quad H_2 = 4, \quad H_3 = 8$$

$$Hf(2,2,2) = \begin{vmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{vmatrix}, \quad H_1 = 2, \quad H_2 = 0, \quad H_3 = -32$$

$$Hf(-2,-2,2) = \begin{vmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix}, \quad H_1 = 2, \quad H_2 = 0, \quad H_3 = -32$$

$$Hf(-2,2,-2) = \begin{vmatrix} 2 & 2 & -2 \\ 2 & 2 & 2 \\ -2 & 2 & 2 \end{vmatrix}, \quad H_1 = 2, \quad H_2 = 0, \quad H_3 = -32$$

$$Hf(2,-2,-2) = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{vmatrix}, \quad H_1 = 2, \quad H_2 = 0, \quad H_3 = -32$$

By the determinant test of positive-/negative-definiteness, we conclude that f attains local minimum at $(0, 0, 0)$, and is of saddle-type at $(2, 2, 2)$, $(-2, -2, 2)$, $(-2, 2, -2)$, and $(2, -2, -2)$. □

Each classification gets +2

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 - y^2$, and S the unit circle in $\mathbb{R}^2$. Find the extrema of $f|_S$ by using the bordered Hessian test. (No credit will be given if there is no use of bordered Hessian test.)

Solution. Let $g(x, y) = x^2 + y^2$. Observe that $\nabla g(x_0, y_0) \neq \overrightarrow{0}$ for all $(x_0, y_0) \in S$. Hence by the Lagrange multiplier theorem, there exists $\lambda \in \mathbb{R}$ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ if $f|_S$ attains a local extremum at (x_0, y_0). That means the following system of equation is satisfied:

$$\begin{aligned} (2x_0, -2y_0) &= \lambda(2x_0, 2y_0) \\ x_0^2 + y_0^2 &= 1. \end{aligned}$$

This gives $(\lambda, x, y) = (1, \pm 1, 0)$ and $(-1, 0, \pm 1)$.

Now form the auxilary function $h(x,y) := f(x,y) - \lambda(g(x,y)-1)$. Note that

$$\begin{array}{ccc} \frac{\partial^2 h}{\partial \lambda^2} = 0 & \frac{\partial^2 h}{\partial x \partial \lambda} = -\frac{\partial g}{\partial x} = -2x & \frac{\partial^2 h}{\partial y \partial \lambda} = -\frac{\partial g}{\partial y} = -2y \\ \frac{\partial^2 h}{\partial \lambda \partial x} = -\frac{\partial g}{\partial x} = -2x & \frac{\partial^2 h}{\partial x^2} = 2-2\lambda & \frac{\partial^2 h}{\partial y \partial x} = 0 \\ \frac{\partial^2 h}{\partial \lambda \partial y} = -\frac{\partial g}{\partial y} = -2y & \frac{\partial^2 h}{\partial x \partial y} = 0 & \frac{\partial^2 h}{\partial y^2} = -2-2\lambda \end{array}$$

So the bordered Hessian determinant $|\overline{H}|$ at (x_0, y_0) is given by

$$\begin{vmatrix} 0 & -2x_0 & -2y_0 \\ -2x_0 & 2-2\lambda & 0 \\ -2y_0 & 0 & -2-2\lambda \end{vmatrix} = 4x_0^2(2+2\lambda) - 4y_0^2(2-2\lambda)$$

Therefore

(λ, x, y)	$\lvert\overline{H}\rvert$	Test
$(1,1,0)$	16	Local maximum
$(1,-1,0)$	16	Local maximum
$(-1,0,1)$	-16	Local minimum
$(-1,0,-1)$	-16	Local minimum

□

Each local extremum gets +2.5

10. Let $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Find the absolute maximum and minimum values of f on the elliptical region $x^2 + \frac{1}{2}y^2 \le 1$.

Solution. We calculate all critical points of f in the open set $\{(x,y) \in \mathbb{R}^2 : x^2 + \frac{1}{2}y^2 < 1\}$. From $f_x = x$ and $f_y = y$, the only critical point is $(x,y) = (0,0)$, and f attains the absolute minimum 0 at $(0,0)$. (Because f is defined by sum of two squares.)

Now let $g(x,y) = x^2 + \frac{1}{2}y^2$. Observe that $\nabla g(x_0, y_0)$ is vanishing only at the origin which is not on the ellipse $S := \{(x,y) \in \mathbb{R}^2 : x^2 + \frac{1}{2}y^2 = 1\}$. By the Lagrange multiplier theorem, there exists $\lambda \in \mathbb{R}$ such that $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$ if $f|_S$ attains a local extremum at (x_0,y_0). That means the following system of equation is satisfied:

$$(x_0, y_0) = \lambda(2x_0, y_0)$$
$$x_0^2 + \frac{1}{2}y_0^2 = 1.$$

This gives $(\lambda, x, y) = (1, 0, \pm\sqrt{2})$ and $(\frac{1}{2}, \pm 1, 0)$. Clearly $f|_S$ attains local maxima 1 at $(0, \pm\sqrt{2})$ and local minima $\frac{1}{2}$ at $(\pm 1, 0)$. Therefore f has its absolute maximum 1 at $(0, \pm\sqrt{2})$. □

Absolute maximum +5, absolute minimum +5.