#1. Step 1: Find all critical points

$\frac{\partial f}{\partial x} = x^2 - x = x(x-1) = 0 \iff x = 0, 1$

$\frac{\partial f}{\partial y} = y^2 - 5y + 6 = (y-2)(y-3) = 0 \iff y = 2, 3.$

Critical points $(0,2)$, $(0,3)$, $(1,2)$, $(1,3)$ Finding all critical points up to 4pts

Step 2 : Calculate Hessian

$\frac{\partial^2 f}{\partial x^2} = 2x - 1$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$ $\frac{\partial^2 f}{\partial y^2} = 2y - 5$ Calculating Hessian up to 2pts

$$Hf(x_0, y_0) = \begin{pmatrix} 2x_0 - 1 & 0 \\ 0 & 2y_0 - 5 \end{pmatrix}$$

(1) $Hf(0,2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (2) $Hf(0,3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det Hf(0,2) > 0$, $Hf(0,2)_{11} < 0$; local maximum. $\det Hf(0,3) < 0$; saddle

(3) $Hf(1,2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (4) $Hf(1,3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det Hf(1,2) < 0$; saddle $\det Hf(1,3) > 0$, $Hf(1,3)_{11} > 0$;

local minimum

Classifying all critical pts up to 4pts.

#2. Let $f(x,y,z) = x$

$$g_1(x,y,z) = x^2+y^2+z^2 \qquad \text{Denote} \qquad S_1 = \{(x,y,z) : g_1 = 1\}$$

$$g_2(x,y,z) = x+y+z \qquad\qquad S_2 = \{(x,y,z) : g_2 = 1\}$$

$$\nabla f = (1,0,0)$$

$$\left.\begin{array}{l}\nabla g_1 = (2x, 2y, 2z) \\ \nabla g_2 = (1,1,1)\end{array}\right\} \text{ linearly independent at } (x,y,z) \in S_1 \cap S_2.$$

If $f\big|_{S_1 \cap S_2}$ attains local extremum at $\vec{x}_0 = (x_0, y_0, z_0)$, there exists λ_1, λ_2 s.t.

$$\nabla f(\vec{x}_0) = \lambda_1 \nabla g_1(\vec{x}_0) + \lambda_2 \nabla g_2(\vec{x}_0).$$

Lagrange multiplier setup 5pts

$$1 = \lambda_1 2x_0 + \lambda_2 \quad -①$$
$$0 = \lambda_1 2y_0 + \lambda_2 \quad -②$$
$$0 = \lambda_1 2z_0 + \lambda_2 \quad -③$$
$$1 = x_0 + y_0 + z_0 \quad -④$$
$$1 = x_0^2 + y_0^2 + z_0^2 \quad -⑤$$

② − ③ $\Rightarrow$ $y_0 = z_0$

$\rightarrow x_0 + 2y_0 = 1$

$\Rightarrow x_0^2 + 2y_0^2 = 1$

$$(1-2y_0)^2 + 2y_0^2 = 1$$
$$1 - 4y_0 + 4y_0^2 + 2y_0^2 = 1$$
$$6y_0^2 - 4y_0 = 0$$

$$(*)\begin{cases} y_0 = 0 \\ x_0 = 1 \end{cases}, \quad (**)\begin{cases} y_0 = \frac{2}{3} \\ x_0 = -\frac{1}{3} \end{cases}.$$

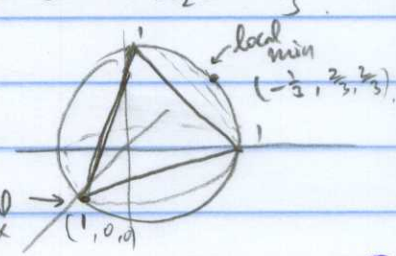
$$\lambda_1 = \frac{1}{2} \qquad \lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = 0 \qquad \lambda_2 = \frac{2}{3}$$

$$(**)\begin{cases} -\frac{2}{3}\lambda_1 + \lambda_2 = 1 \\ \frac{4}{3}\lambda_1 + \lambda_2 = 0 \end{cases}$$

$$-2\lambda_1 = 1$$

$$\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \frac{2}{3}.$$

$$(*)\begin{cases} 1 = 2\lambda_1 + \lambda_2 \\ 0 = \lambda_2 \end{cases}$$



At $(1,0,0)$, $f\big|_{S_1 \cap S_2}$ attains a local maximum.

At $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, $f\big|_{S_1 \cap S_2}$ attains a local minimum.

Each correct answer 2.5 pts.

#3. False.

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$ Take $\vec{x}_0 = (0,0)$.

First note that this function is continuous at $(0,0)$.
This should be proved. For a proof, please refer PDF file uploaded at 2016-03-02 announcement.

Correct example
Addressing that the given function is continuous 5pts.

Partial derivatives of f at $(0,0)$ exists.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

partial derivatives at $\vec{x}_0$ exist
2pts

However f is not differentiable at $(0,0)$, because

$$\lim_{\vec{h}=(h_1,h_2)\to(0,0)} \frac{|f(\vec{0}+\vec{h}) - f(0) - \nabla f(0,0)\cdot\vec{h}|}{\|\vec{h}\|} = \lim_{(h_1,h_2)\to(0,0)} \frac{\left|\frac{h_1 h_2}{\sqrt{h_1^2+h_2^2}}\right|}{(h_1^2+h_2^2)^{1/2}}$$

f is not differentiable at $\vec{x}_0$.
3pts

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{|h_1 h_2|}{h_1^2+h_2^2},$$ and this limit does not exist.

Approaching to $(0,0)$ along the diagonal yields $\frac{1}{2}$ as the limit whereas 0 when approaching along the x- or the y-axis.

#4. If $\|\vec{r}(t)\|$ attains a local extremum, so does $\|\vec{r}(t)\|^2 = \vec{r}(t)\cdot\vec{r}(t)$.

Let $\|\vec{r}(t_0)\|^2$ be a local extremum. From

$$0 = \frac{d}{dt}\|\vec{r}(t_0)\|^2 = \vec{r}\,'(t_0)\cdot\vec{r}(t_0) + \vec{r}(t_0)\cdot\vec{r}\,'(t_0) = 2\,\vec{r}(t_0)\cdot\vec{r}\,'(t_0),$$

it follows that

$$\vec{r}(t_0) \perp \vec{r}\,'(t_0).$$

#5. See the textbook p.277 Theorem 3. Statement 5 proof 5.

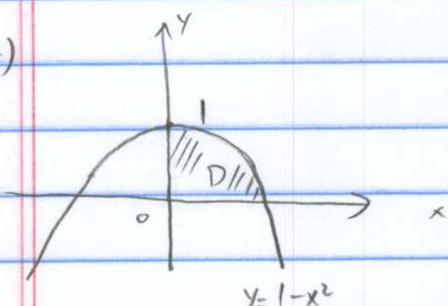
#6 (1)

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos nx \sin my \, dx\, dy = \left(\int_{-\pi}^{\pi}\cos nx\, dx\right)\left(\cancel{\int_{-\pi}^{\pi}\sin my\, dy}\right) = 0$$

$\to 0$

Sine is an odd function.

(2)



$$\iint_D (x^2 + xy - y^2)\, dA$$

$$= \int_0^1\int_0^{1-x^2} x^2+xy-y^2 \, dy\, dx$$

Correct setup +4 pts

$$= \int_0^1 x^2y + \tfrac{1}{2}xy^2 - \tfrac{1}{3}y^3 \Big|_0^{1-x^2} dx$$

$$= \int_0^1 x^2(1-x^2) + \tfrac{1}{2}x(1-x^2)^2 - \tfrac{1}{3}(1-x^2)^3 \, dx$$

$$= \int_0^1 x^2 - x^4 + \tfrac{1}{2}x(1-2x^2+x^4) - \tfrac{1}{3}(1-3x^2+3x^4-x^6)\, dx$$

$$= \int_0^1 2x^2 - 2x^4 + \tfrac{1}{2}x - x^3 + \tfrac{1}{2}x^5 - \tfrac{1}{3} + \tfrac{x^6}{3}\, dx$$

$$= \tfrac{2}{3}x^3 - \tfrac{2}{5}x^5 + \tfrac{1}{4}x^2 - \tfrac{1}{4}x^4 + \tfrac{1}{12}x^6 - \tfrac{1}{3}x + \tfrac{1}{21}x^7 \Big|_0^1$$

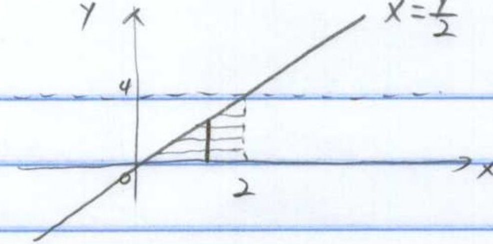
$$= \tfrac{2}{3} - \tfrac{2}{5} + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{4}} + \tfrac{1}{12} - \tfrac{1}{3} + \tfrac{1}{21}$$

$$= \tfrac{1}{3} - \tfrac{2}{5} + \tfrac{1}{12} + \tfrac{1}{21} = \frac{7\times4\times5 - 2\times7\times3\times4 + 35 + 20}{21\times20} = \frac{27}{21\times20}$$

$$= \frac{9}{140}$$

Correct answer 1 pt.

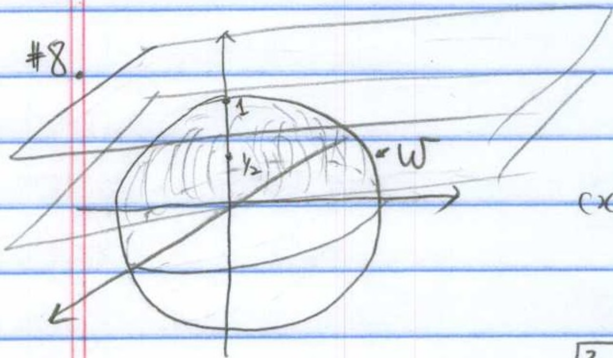
#7. $\int_0^4 \int_{y/2}^{2} e^{x^2}\,dx\,dy$



$= \int_0^2 \int_0^{2x} e^{x^2}\,dy\,dx$

$= \int_0^2 e^{x^2} y \Big|_0^{2x} dx = \int_0^2 2x e^{x^2} dx = e^{x^2}\Big|_0^2 = \underline{e^4 - 1}$

#8.



$W = \{(x,y,z) \in \mathbb{R}^3 : \frac{1}{2} \le z \le 1 \text{ and } x^2+y^2+z^2 \le 1\}$

$(x,y,z) \in W$ satisfies

$\frac{1}{2} \le z \le \sqrt{1-x^2-y^2}$

$-\sqrt{\frac{3}{4}-x^2} \le y \le \sqrt{\frac{3}{4}-x^2}$

$x^2 + (\frac{1}{2})^2 = 1$

$x^2 = \frac{3}{4}$

$x = \pm \frac{\sqrt{3}}{2}$

$-\frac{\sqrt{3}}{2} \le x \le \frac{\sqrt{3}}{2}$

Error in upper/lower ends of integral -2

Volume $= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}} \int_{\frac{1}{2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$

Correct integral $+10$

$\#9. \ I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Proof:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} \lim_{R\to\infty}\int_0^R -2re^{-r^2}\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} \lim_{R\to\infty} e^{-r^2}\Big|_0^R\,d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} \lim_{R\to\infty} e^{-R^2} - 1\,d\theta = \pi.$$

Note that $LHS = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)$.

So $I^2 = \pi, \quad I = \sqrt{\pi}$.

+5pts.

Now $\int_{-\infty}^{\infty} e^{-ax^2}dx = \int_{-\infty}^{\infty} e^{-\sqrt{a}x\cdot\sqrt{a}x}\frac{1}{\sqrt{a}}d(\sqrt{a}x) = \frac{1}{\sqrt{a}}\sqrt{\pi} = \underline{\underline{\sqrt{\frac{\pi}{a}}}}$.

+5pts

$\#10. \ I_y = \iiint_{\mathbb{B}_R} (x^2+z^2)\mu \,dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} (\rho^2\sin^2\phi\cos^2\theta + \rho^2\cos^2\phi)\mu\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$$= \int_0^{\pi}\int_0^{2\pi} \mu\cdot\frac{1}{5}\rho^5(\sin^3\phi\cos^2\theta + \cos^2\phi\sin\phi)\Big|_0^R d\theta\,d\phi$$

Partial Credits up to +5 for correct triple integral setup.

$$= \int_0^{\pi}\frac{\mu R^5}{5}\left[\sin^3\phi\int_0^{2\pi}\cos^2\theta\,d\theta + 2\pi\cos^2\phi\sin\phi\right]d\phi$$

$$= \frac{\mu R^5}{5}\int_0^{\pi}\sin^3\phi\left(\frac{1}{2}\theta + \cancel{\frac{1}{4}\sin 2\theta}\Big|_0^{2\pi}\right) + 2\pi\cos^2\phi\sin\phi\,d\phi$$

Expression with total mass is also fine.

$$= \frac{\mu R^5}{5}\pi\left[\underbrace{\int_0^{\pi}\sin^3\phi\,d\phi}_{\int_1^{-1}(x^2-1)dx = 4/3} + \underbrace{\int_0^{\pi}2\cos^2\phi\sin\phi\,d\phi}_{\int_{-1}^{1}2y^2dy = \frac{4}{3}}\right] = \underline{\underline{\frac{8}{15}\cdot\mu R^5\pi}}.$$