1. Prove the following result: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$

Solution. Expand the given determinant along the first column. We get

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 \\ c & c^2 \end{vmatrix} + \begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix} = bc^2 - cb^2 - (ac^2 - ca^2) + ab^2 - ba^2.$$

By rearranging terms, we see the far RHS is equal to the following. Factor out $(b-c)$.

$$bc^2 - cb^2 + ab^2 - ac^2 - (ba^2 - ca^2) = -bc(b-c) + a(b+c)(b-c) - a^2(b-c)$$
$$= -(b-c)\big(a^2 - (b+c)a + bc\big) = -(b-c)(a-b)(a-c) = (a-b)(b-c)(c-a).$$

□