5. (1) Decide if each of the given functions is differentiable.

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}.$$

Solution. Let $(x_0, y_0) \neq (0,0)$. We prove that f is differentiable at (x_0, y_0). The idea is using the following proposition (you can use it without giving a proof).

Proposition 1. If f is of class C^1 at (x_0, y_0) (i.e., f has all first partial derivatives at (x_0, y_0) and the partial derivatives are continuous at at (x_0, y_0)), then f is differentiable.

We claim that our f is of class C^1 at $(x_0, y_0) \neq (0,0)$, since it has partial derivatives

$$\text{(1)} \qquad \begin{aligned} \frac{\partial f}{\partial x} &= \frac{y\sqrt{x^2+y^2} - xy\frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2y + y^3 - x^2y}{(x^2+y^2)^{3/2}} \\ \frac{\partial f}{\partial y} &= \frac{x\sqrt{x^2+y^2} - xy\frac{2y}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{xy^2 + x^3 - xy^2}{(x^2+y^2)^{3/2}} \end{aligned}$$

and these are continuous functions at $(x_0, y_0) \neq (0,0)$. Therefore, by the Proposition 1, differentiability of f follows.

Now we consider the case when $(x_0, y_0) = (0,0)$. Since the function is not defined at $(0,0)$, f is not differentiable at that point. (Notice that to even discuss differentiability of f at $(0,0)$ we should know $f((0,0))$ to verify the definition.) □

Remark. The above problem becomes more interesting if we change the problem as follows.

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x_0, y_0) \neq (0,0) \\ 0 & \text{if } (x_0, y_0) = (0,0) \end{cases}$$

Solution. In the case $(x_0, y_0) \neq (0,0)$, the first two paragraphs of the above solution identically becomes a solution.

Now let $(x_0, y_0) = (0,0)$. We first calculate partial derivatives of f at $(0,0)$ if exists. (Notice that you cannot plug in $(0,0)$ to the equation (1), and you have to go to the definition of the partial differentiation.)

$$\text{(2)} \qquad \begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(0,0)} &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \\ \left.\frac{\partial f}{\partial y}\right|_{(0,0)} &= \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \end{aligned}$$

Now we check f has a good approximation at $(0,0)$. In other words, we check if the following equality holds:

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f((0,0)+(h_1,h_2)) - f((0,0)) - \nabla f((0,0))\cdot(h_1,h_2)|}{\|(h_1,h_2)\|} = 0.$$

We see that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f((h_1,h_2)) - 0 - 0|}{\|(h_1,h_2)\|} = \lim_{(h_1,h_2)\to(0,0)} \frac{\left|\frac{h_1h_2}{\sqrt{{h_1}^2+{h_2}^2}}\right|}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{h_1h_2}{h_1^2+h_2^2}.$$

We claim that the far RHS is not zero, because if we let $h_1 = r\cos\theta$ and $h_2 = r\sin\theta$, it becomes $\lim_{r\to 0}\frac{|r\cos\theta r\sin\theta|}{r^2} = \left|\frac{1}{2}\sin 2\theta\right|$. It means as long as we approach to the origin any straight line that is not the coordinate axis, the limit is non-zero, whereas it is zero when we approach along one of the coordinate axis. Therefore, we conclude that f is not differentiable at $(0,0)$.

Remark (Not a part of the solution)**:** Notice that partial derivatives f_x and f_y at $(0,0)$ are not continuous: In equation (1), $\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial x}$ and $\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial y}$ do not exist. (Approaching along x-axis and y-axis yield different values.) So limits do not exist, and hence cannot be continuous. This means we cannot apply Proposition 1 and have to proceed as above. □

A sketch of solution for Problem 5. (2): We are given the following function.

$$f(x,y) = \frac{2xy}{(x^2+y^2)^2}.$$

This function is differentiable at all $(x_0,y_0)\neq(0,0)$ by an argument similar to Problem 5. (1), and is not defined (and hence not differentiable) at $(0,0)$.

Now consider the following variant of this problem:

$$f(x,y) = \begin{cases} \frac{2xy}{(x^2+y^2)^2} & \text{if } (x_0,y_0)\neq(0,0) \\ 0 & \text{if } (x_0,y_0)=(0,0) \end{cases}$$

I claim that the function is not differentiable at $(0,0)$. The function has partial derivatives at $(0,0)$ as $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0$ and $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$. (You check these yourself if not convincing.) We now compute the following limit:

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f((0,0)+(h_1,h_2))-f((0,0))-\nabla f((0,0))\cdot(h_1,h_2)|}{\|(h_1,h_2)\|} = \lim_{(h_1,h_2)\to(0,0)} \frac{\left|\frac{2h_1h_2}{({h_1}^2+{h_2}^2)^2}\right|}{({h_1}^2+{h_2}^2)^{1/2}}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{|2h_1h_2|}{({h_1}^2+{h_2}^2)^{5/2}}.$$

We let $h_1 = r\cos\theta$ and $h_2 = r\sin\theta$. Then the far RHS becomes

$$\lim_{r\to 0} \frac{|\sin 2\theta|}{r^3}.$$

This limit certainly is not 0.