9. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable at $\overrightarrow{x}_0 \in \mathbb{R}^3$. Prove that

$$\frac{|f(\overrightarrow{x}) - f(\overrightarrow{x}_0)|}{\|\overrightarrow{x} - \overrightarrow{x}_0\|}$$

is bounded by a positive constant. (Hint: Use the triangle inequality and the Cauchy-Schwarz inequality)

Solution. Recall the following inequalities:

$$\|\overrightarrow{v}\| + \|\overrightarrow{w}\| \le \|\overrightarrow{v} + \overrightarrow{w}\| \qquad \text{for any } \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n. \tag{1}$$

called the triangle inequality, and

$$|\overrightarrow{v} \cdot \overrightarrow{w}| \le \|\overrightarrow{v}\|\|\overrightarrow{w}\| \qquad \text{for any } \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n. \tag{2}$$

which is called the Cauchy-Schwarz inequality. Note that the following inequality is nothing but a restatement of (1).

$$\|\overrightarrow{v}\| - \|\overrightarrow{w}\| \le \|\overrightarrow{v} - \overrightarrow{w}\| \qquad \text{for any } \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n. \tag{3}$$

This is because $\|\overrightarrow{v}\| = \|\overrightarrow{v} - \overrightarrow{w} + \overrightarrow{w}\|$.

Now we prove the given statement using the above inequalities. Recall that f is differentiable at $\overrightarrow{x}_0$ if partial derivatives $f_{x_1}(\overrightarrow{x}_0), \cdots, f_{x_n}(\overrightarrow{x}_0)$ exists and

$$\lim_{\overrightarrow{h} \to \overrightarrow{0}} \frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0) - \nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} = 0.$$

Recall the $\epsilon - \delta$ definition of limit. (Section 2.2.) Having the above limit *implies* that there exists some $\delta > 0$ such that $0 < \|\overrightarrow{h}\| < \delta$ implies $\left| \frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0) - \nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} - 0 \right| < 1$. (Note: We chose $\varepsilon = 1$ which we can.)

Therefore we look at

$$\frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0) - \nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} < 1.$$

By (3), we observe that

$$\frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0)| - |\nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} \le \frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0) - \nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|} < 1$$

Hence

$$\frac{|f((\overrightarrow{x}_0 + \overrightarrow{h}) - f(\overrightarrow{x}_0)|}{\|\overrightarrow{h}\|} < 1 + \frac{|\nabla f(\overrightarrow{x}_0) \cdot \overrightarrow{h}|}{\|\overrightarrow{h}\|},$$

and notice that by (2),

$$1+\frac{|\nabla f(\overrightarrow{x}_0)\cdot\overrightarrow{h}|}{\|\overrightarrow{h}\|}\le 1+\frac{\|\nabla f(\overrightarrow{x}_0)\|\|\overrightarrow{h}\|}{\|\overrightarrow{h}\|}=1+\|\nabla f(\overrightarrow{x}_0)\|$$

where the far RHS is a constant that is positive. □