Exam I
Spring 2017 MATH 15500 Section 06
March 7th, 2017. 9:10AM–11:00AM

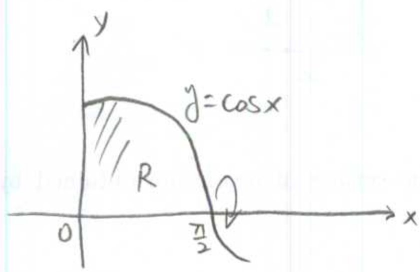
Your name:

Instructions: Please clearly write your name above. This exam is closed-book and closed-note. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Write all details explicitly. Answers without justifications and/or calculation steps may receive no score. Hand-in both the exam sheet and your work on given sheets.

Total 100 points. 10 points each.

Calculation mistakes and other minor mess-up: −1

1. Let R be the region bounded by the x-axis, y-axis, and the function $y = \cos x$. Find the volume of the solid generated when R is revolved about the x-axis. (Hint: $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$)



$$V = \int_0^{\frac{\pi}{2}} \pi y^2 \, dx = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

Correct Setup +5

$$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2x)\, dx = \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

Correct integral +3

$$= \frac{\pi}{2}\left(\left(\frac{\pi}{2}+0\right) - (0+0)\right) = \frac{\pi^2}{4}.$$

(*): Many of you got it incorrect!

2. Find the arc length of the curve given by the function

$$y = \frac{e^x + e^{-x}}{2}$$

on $[-\ln 2, \ln 2]$ by integrating with respect to x.

(*)

$$L = \int_{-\ln 2}^{\ln 2} \sqrt{1+(y')^2}\, dx = \int_{-\ln 2}^{\ln 2} \sqrt{1+\left(\frac{e^x - e^{-x}}{2}\right)^2}\, dx$$

+2pts up to here

+5pts more

$$= \int_{-\ln 2}^{\ln 2} \sqrt{1 + \frac{e^{2x}}{4} + \frac{e^{-2x}}{4} - \frac{1}{2}}\, dx = \int_{-\ln 2}^{\ln 2} \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx$$

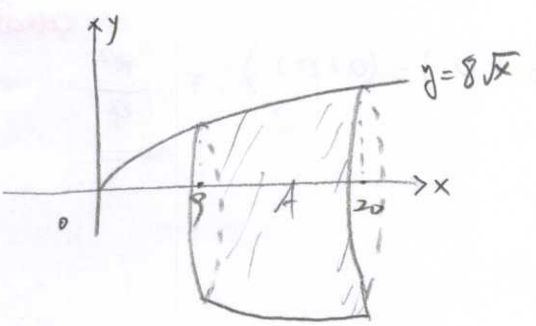
total 7 pts up to here

(*)

$$= \left[\frac{e^x - e^{-x}}{2}\right]_{-\ln 2}^{\ln 2} = \frac{2-\frac{1}{2}}{2} - \frac{\frac{1}{2}-2}{2} = \frac{3}{2}$$

total 9 pts up to here

3. For the function $y = 8\sqrt{x}$ on $[9, 20]$, find the area of the surface of revolution obtained by revolving the graph about x-axis.



$$A = \int_9^{20} 2\pi y \sqrt{1+(y')^2}\, dx$$ +2pts

$$= \int_9^{20} 16\pi\sqrt{x} \cdot \sqrt{1+\frac{16}{x}}\, dx$$ +3pts

$$= 16\pi \int_9^{20} \sqrt{x+16}\, d(x+16)$$

$$= 16\pi \cdot \frac{2}{3}\left[(x+16)^{3/2}\right]_9^{20} = \frac{32\pi}{3}\left(36^{3/2} - 25^{3/2}\right)$$

$$= \frac{32}{3}\pi\left(6^3 - 5^3\right) = \frac{32}{3}\pi(216-125) = \frac{2912}{3}\pi$$

4. Suppose a force of 30N is required to stretch and hold a spring 0.3m from its equilibrium position. How much additional work is required to compress the spring 0.2m if it has already been compressed 0.3m from its equilibrium?

Spring Constant k: $F = kx$

$30N = k \cdot 0.3\ m$

$\therefore\ k = 100\ N/m$

Correct k : 3pts

Additional Work $= \int_{-0.3}^{-0.5} kx\, dx = \frac{1}{2}kx^2 \Big|_{-0.3}^{-0.5}$

Partial Credits on Correct integral Setup.

Correct integral 3pts.

$= 50 \cdot (0.25 - 0.09) = 100 \cdot 0.08 = \underline{8\ J}$

3pts

5. For the function $f(x) = x^3 - 4$, find the slope of the tangent line on the point $\overset{(4,2)}{\cancel{(2,4)}}$ of f^{-1}.

$$y = x^3 - 4 \quad , \quad \left.\frac{dy}{dx}\right|_{(2,4)} = 3x^2\Big|_{x=2} = 12 .$$

+3pts

$$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=4} = \left.\frac{dx}{dy}\right|_{(4,2)} = \frac{1}{\left.\frac{dy}{dx}\right|_{(2,4)}} = \frac{1}{12}$$

+ 5pts

6. Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} \frac{1 + \cos x}{x + \sin x} dx.$$

Note that $\frac{d}{dx}(x + \sin x) = 1 + \cos x$.

$$\int_0^{\pi/2} \frac{1+\cos x}{x + \sin x} dx = \ln |x + \sin x| \Big|_0^{\pi/2} = \ln\left(\frac{\pi}{2} + 1\right) - \ln 0$$

10 pts up to here

$$= \infty$$

7. Calculate the following integral:

$$\int \frac{1}{25x^2+1}dx.$$

$$\int \frac{dx}{25x^2+1} = \int \frac{\frac{1}{5}d(5x)}{(5x)^2+1} = \frac{1}{5}\int \frac{1}{(5x)^2+1}\,d(5x) = \frac{1}{5}\tan^{-1}5x + C.$$

No Integration Const: −1pt

No partial credit otherwise.

If Correct answer without enough justification: 5pts.

8. Evaluate the limit:

$$\lim_{x\to 0^+} (\csc x)^x$$

Note that the given limit is ∞^0-form.

$$(\csc x)^x = e^{x\ln\csc x} = e^{-\frac{\ln\sin x}{1/x}}$$ where $\ln\sin x \to -\infty$, $1/x \to \infty$ as $x \to 0^+$.

+4pts

$$\lim_{x\to 0^+}(\csc x)^x = e^{-\lim_{x\to 0^+}\frac{\ln\sin x}{1/x}} \overset{\text{L'Hôpital's rule}}{=} e^{-\lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}}$$ +4pts

$$= e^{\lim_{x\to 0^+}\underbrace{\frac{x}{\sin x}}_{\to 1}\cdot \underbrace{x}_{\to 0}\cdot\underbrace{\cos x}_{\to 1}} = e^0 = \underline{1}.$$

Note: $\lim_{x\to 0}\frac{x}{\sin x} = 1$.

9. Calculate the following integral:

$$\int \frac{\sin x + \tan x}{\cos^2 x} dx.$$

Each Correct integral 4pts. Integration Const: 1pt. No partial Credits otherwise

$$\int \frac{\sin x + \tan x}{\cos^2 x} dx = \int \frac{\sin x}{\cos^2 x} dx + \int \frac{\sin x}{\cos^3 x} dx$$

$$\underset{t=\cos x}{=} -\int \frac{dt}{t^2} - \int \frac{dt}{t^3} = -\frac{1}{1-2} t^{1-2} - \frac{1}{1-3} t^{1-3} + C$$

$$= t^{-1} + \frac{1}{2} t^{-2} + C$$

$$= \frac{1}{\cos x} + \frac{1}{2\cos^2 x} + C$$

10. Calculate the following integral:

$$\int \ln x \quad dx.$$

$$\int \ln x \quad dx = \ln x \cdot x - \int \frac{1}{x} \cdot x \, dx = x \ln x - x + C$$

$u = \ln x \quad dv = 1$

$du = 1/x \quad v = x$

Correct answer without enough justification: 5pts

Knowing and Correct application of integration by parts: 5pts

each term for the integral 2pts

Integration Constant 1pt