Exam II
Spring 2017 MATH 15500 Section 06
April 4th, 2017. 09:00–11:00

Your name:

Instructions: Please clearly write your name above. This exam is closed-book and closed-note. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Write all details explicitly. Answers without justifications and/or calculation steps may receive no score.

Total 100 points. 10 points each unless specified otherwise. Minor mistakes: −1 ~ −3 pts. Such as Sign depending on Significance.

1. Calculate the following integral:

$$\int \sec^3 \theta d\theta$$

Hint: You may use $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$.

$$\int \sec^3\theta\, d\theta = \int \sec^2\theta \sec\theta\, d\theta = \tan\theta\sec\theta - \int \tan\theta \sec\theta\tan\theta\, d\theta$$

$$= \tan\theta\sec\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$$

$$= \tan\theta\sec\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

$$2\int \sec^3\theta\, d\theta = \tan\theta\sec\theta + \ln|\sec\theta + \tan\theta| + C$$

Hence $\underline{\int \sec^3\theta\, d\theta = \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C}$.

No partial credit if you get an incorrect answer through the reduction formula

2. Calculate the following integral:

$$\int \cos^2 x \sin^2 x dx$$

$$\left(\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}. \right) \cdots (*)$$

$$\cos^2 x \sin^2 x = \frac{(1+\cos 2x)(1-\cos 2x)}{4} = \frac{1-\cos^2 2x}{4}. = \frac{1}{4} - \frac{1}{4} \cdot \frac{1+\cos 4x}{2}.$$

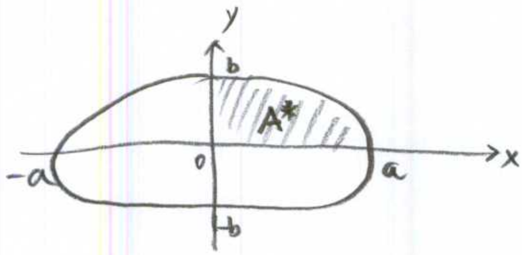
Hence $\int \cos^2 x \sin^2 x \, dx = \int \frac{1}{4} - \frac{1}{4} \cdot \frac{1}{2}(1+\cos 4x)\, dx$

2 pts up to this point (Correctly using (*) ~~abo~~ above.)

$$= \frac{1}{4}x - \frac{1}{8}\left[x + \frac{1}{4}\sin 4x\right] + C$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C.$$

3. Prove that the area of an ellipse, whose equation is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is $ab\pi$.



Area $= 4A^*$.

$$A^* = \int_0^a y\,dx = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

4 points up to here

Put $x = a\sin\theta$ (x: $0 \to a$, θ: $0 \to \frac{\pi}{2}$)

$dx = a\cos\theta\,d\theta$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Leftrightarrow b^2x^2 + a^2y^2 = a^2b^2$$

$$a^2y^2 = a^2b^2 - b^2x^2$$

$$y = \frac{\pm b\sqrt{a^2 - x^2}}{a}$$

2 points up to here

$$= \frac{b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\,d\theta$$

$$= \frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta \cdot a\cos\theta\,d\theta$$

7 points up to here

$$= ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= ab\left[\frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}ab.$$

Hence the area of ellipse is πab.

4. Calculate

$$\int \frac{x^2}{\sqrt{16-x^2}}dx.$$

Let $x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$$\int \frac{x^2}{\sqrt{16-x^2}}dx = \int \frac{16\sin^2\theta \cdot 4\cos\theta}{\sqrt{16\cos^2\theta}}d\theta$$

Correct Sine Substitution 4 pts

$$= \int 16 \cdot \frac{1-\cos 2\theta}{2} d\theta = 8\int 1 - \cos 2\theta\, d\theta$$

$$= 8\left[\theta - \frac{1}{2}\sin 2\theta\right] = 8\theta - 4\sin 2\theta + C$$

Correct integral with variable θ 2 pts

$$\sin^{-1}\frac{x}{4} = \theta$$

(Note $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$)

$$\sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \frac{x}{4} \cdot \frac{\sqrt{16-x^2}}{4}$$

This step 3pts

$$= 8\sin^{-1}\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2} + C$$

5. Calculate

$$\int \frac{dx}{x^2-3x-4}.$$

$$x^2-3x-4 = (x-4)(x+1)$$

$$\int \frac{dx}{x^2-3x-4} = \int \frac{dx}{(x-4)(x+1)} = \int \frac{1}{5}\left(\frac{1}{x-4} - \frac{1}{x+1}\right)dx$$

- No partial credit for incorrect partial fraction
- Correct partial fraction: 5pts

$$= \frac{1}{5}\left(\ln|x-4| - \ln|x+1|\right) + C$$

Any improper integral without explicit distinction that it is limit of finite integral : -3 pts

6. (5 points each) Let $f(x) = \frac{1}{x^p}$, where $0 < p < \infty$. Discuss the convergence of the definite integral $\int_1^\infty f(x)dx$ in the following cases:

(1) When $0 < p < 1$:

$$\int_1^\infty f(x)dx = \lim_{M\to\infty}\int_1^M \frac{1}{x^p}\,dx = \lim_{M\to\infty} \frac{1}{1+(-p)} x^{1-p}\Big|_1^M$$

$$= \lim_{M\to\infty} \frac{1}{1-p}\left(M^{1-p} - 1\right) = (*)$$

Since $0 < 1-p < 1$, $\lim_{M\to\infty} M^{1-p} = \infty$

Therefore $(*) = \infty$. The given improper integral diverges.

(2) When $p = 1$:

$$\int_1^\infty f(x)dx = \lim_{M\to\infty}\int_1^M \frac{1}{x}dx = \lim_{M\to\infty} \ln|x|\Big|_1^M = \lim_{M\to\infty}\ln(M) - \ln 1 = \infty$$

The given improper integral diverges.

(3) When $p > 1$:

$$\int_1^\infty f(x)dx = \lim_{M\to\infty}\int_1^M \frac{1}{x^p}dx = \lim_{M\to\infty}\frac{M^{1-p}-1}{1-p} = (**)$$

Since $1-p < 0$

$\lim_{M\to\infty}\left(\frac{1}{M}\right)^{p-1} = 0$. Hence $(**) = \frac{1}{p-1}$.

The given improper integral converges to $\frac{1}{p-1}$.

7. Find the constant k that satisfies the following equation:

$$\int_{-\infty}^{\infty} \frac{k}{4+x^2} dx = 1.$$

$$LHS = \lim_{M\to\infty} \int_{-M}^{M} \frac{k}{4+x^2} dx = k \lim_{M\to\infty} \int_{-M}^{M} \frac{\frac{1}{4}}{1+(\frac{x}{2})^2} d(\tfrac{x}{2}\cdot 2)$$

$$= \frac{k}{2} \lim_{M\to\infty} \int_{-M}^{M} \frac{d(\frac{x}{2})}{1+(\frac{x}{2})^2} = \frac{k}{2} \lim_{M\to\infty} \tan^{-1}\frac{x}{2}\Big|_{-M}^{M} = \frac{k}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi k}{2}$$

So $\frac{\pi k}{2} = 1$, and $k = \frac{2}{\pi}$.

No partial credit.

8. Find the value that the following infinite sum converges to:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

$$\sum_{k=2}^{\infty} = \lim_{N\to\infty} \sum_{k=2}^{N} \frac{1}{(n-1)(n+1)} = \frac{1}{2} \lim_{N\to\infty} \sum_{k=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\Big[\frac{1}{2-1} - \cancel{\frac{1}{2+1}} + \frac{1}{3-1} - \cancel{\frac{1}{3+1}} + \cancel{\frac{1}{4-1}} - \cancel{\frac{1}{4+1}} + \cdots + \cancel{\frac{1}{(N-1)-1}} - \frac{1}{(N-1)+1} + \cancel{\frac{1}{N-1}} - \frac{1}{N+1} \Big]$$

$$= \lim_{N\to\infty}\left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2N} - \frac{1}{2(N+1)}\right] = \underline{\underline{\frac{3}{4}}}.$$

No partial credit.

9. Show that the following sequence converges and find the limit.

$$a_n = \frac{(-1)^n}{n!}.$$

Here $n! := n \cdot (n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$.

No partial Credit.

$$-\frac{1}{n!} \leq a_n = \frac{(-1)^n}{n!} \leq \frac{1}{n!}$$

Since $\lim_{n\to\infty} \frac{1}{n!} = 0 = \lim_{n\to\infty} -\frac{1}{n!}$,

By the Squeeze theorem, $\lim_{n\to\infty} a_n$ converges and the limit is 0.

10 (5 points). Evaluate the following geometric series:

$$1 + \frac{e}{\pi} + \frac{e^2}{\pi^2} + \ldots + \frac{e^n}{\pi^n} + \ldots.$$

This is a geometric Series with
First term $a = 1$
ratio $r = \frac{e}{\pi}$.

No partial Credit.

So the Sum is $\frac{a}{1-r} = \frac{1}{1-\frac{e}{\pi}} = \frac{\pi}{\pi - e}$.