Final Examination
Spring 2017 MATH 15500 Section 06
May 19th, 2017. 09:00–11:00

Your name:

Instructions: Please clearly write your name above. This exam is closed-book and closed-note. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Write all details explicitly. Answers without justifications and/or calculation steps may receive no score.

Total 100 points. 10 points each unless specified otherwise.

1. (8 points) The graph of $f(x) = 2\sqrt{x}$ on the interval $[1, 3]$ is revolved about the x-axis. What is the area of surface generated?

$$A = \int_1^3 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx.$$

$f(x) = 2\sqrt{x}$

$f'(x) = \frac{1}{\sqrt{x}}$

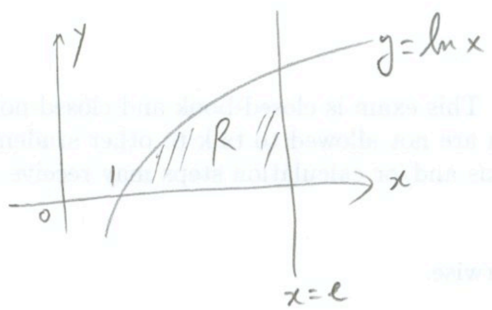
$1+(f'(x))^2 = 1 + \frac{1}{x}$

$$= \int_1^3 2\pi \cdot 2\sqrt{x} \cdot \sqrt{1+\frac{1}{x}}\, dx$$

$$= 4\pi \int_1^3 \sqrt{x+1}\, dx \qquad {}^{4}_{2}\rangle u = x+1 \langle^{3}_{1}$$

$$= 4\pi \left[\frac{2}{3} u^{3/2} \Big|_2^4 \right] = \underline{\frac{8\pi}{3}(8 - 2\sqrt{2})}$$

2. Let R be the region bounded by $y = \ln x$, the x-axis, and the line $x = e$. Find the volume of the solid generated when the region R is revolved about the x-axis.



$$V = \int_1^e \pi (\ln x)^2 \, dx$$

$$= \pi \left[(\ln x)^2 \cdot x \Big|_1^e - \int_1^e \frac{2 \ln x}{x} \cdot x \, dx \right]$$

$$= \pi \left[e - \int_1^e 2 \ln x \, dx \right]$$

$$= \pi e - 2\pi \left[x \ln x - x \right]_1^e$$

$$= \pi e - \cancel{2\pi e} + \cancel{2\pi e} + 2\pi$$

$$= \underline{\pi (e+2)}$$

3. (5 points each) Evaluate or show divergence:

(1)

$$\int_{-\infty}^{\infty} \frac{1}{x^2+9}dx.$$

$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+9} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1/9}{\left(\frac{x}{3}\right)^2+1} \cdot 3\, d\left(\frac{x}{3}\right)$$

$$= \frac{1}{3} \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{\left(\frac{x}{3}\right)^2+1} d\left(\frac{x}{3}\right) = \frac{1}{3} \lim_{R\to\infty} \tan^{-1}\left(\frac{x}{3}\right) \Big|_{-R}^{R}$$

$$= \frac{1}{3} \lim_{R\to\infty} \left(\tan^{-1}\left(\frac{R}{3}\right) - \tan^{-1}\left(\frac{-R}{3}\right)\right) = \frac{1}{3}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{\pi}{3}$$

(2)

$$\int_{0}^{\infty} \frac{e^{2x}}{e^{2x}+1}dx.$$

$$f(x) = e^{2x}+1$$
$$f'(x) = 2e^{2x}$$

$$= \frac{1}{2}\int_0^{\infty} \frac{(e^{2x}+1)'}{(e^{2x}+1)} dx = \lim_{R\to\infty} \frac{1}{2}\int_0^R \frac{(e^{2x}+1)'}{e^{2x}+1} dx$$

$$= \frac{1}{2} \lim_{R\to\infty} \ln\left(e^{2x}+1\right)\Big|_0^R = \infty.$$

4. (3 points each) Compute the limit of the sequence or show divergence:
(1)

$$\lim_{k\to\infty} \frac{e^k}{k^2}.$$

L'Hôpital's rule

$$= \lim_{k\to\infty} \frac{e^k}{2k}$$

$$= \lim_{k\to\infty} \frac{e^k}{2} = \infty$$

(2)

$$\lim_{n\to\infty} \frac{2\sin n^2}{n^3}.$$

$$\frac{-2}{n^3} \le \frac{2\sin n^2}{n^3} \le \frac{2}{n^3}$$

$\frac{-2}{n^3} \to 0$ as $n\to\infty$, $\frac{2}{n^3} \to 0$ as $n\to\infty$.

By Squeeze theorem, $\lim_{n\to\infty} \frac{2\sin n^2}{n^3} = 0$.

(3)

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k.$$

$$= 1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \cdots$$

Geometric Series with ratio $\frac{3}{2} > 1$.

The limit does not exist.

5. Given an infinite series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}},$$

show that the series is divergent using indicated methods:

(1) (3 points) The comparison test. (You can use $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent without proof.)

$$\frac{1}{\sqrt{n-1}} \geqslant \frac{1}{\sqrt{n}}.$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} \geqslant \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} : \text{divergent}$$

So by Comparison test, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$: divergent.

(2) (7 points) The integral test.

$$\int_2^{\infty} \frac{1}{\sqrt{x-1}}\, dx \underset{\substack{\uparrow \\ {}_{1}^{\infty} > u = x-1 < {}_{2}^{\infty}}}{=} \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big|_1^{\infty} = \infty.$$

Hence by the integral test, the given Series diverges.

6. Determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}.$$

Let $a_n = \frac{\ln n}{n^2}$, $b_n = n^{-3/2}$.

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$$ ($\because$ By L'Hopital's rule or Considering the growth rate of $\ln n$ is slower than that of $\sqrt{n}$).

Since $\sum b_n$: Convergent (p-series with $p = 3/2$), by the Limit Comparison test, $\sum a_n$ Converges.

7. Show that the following series is absolutely convergent, convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}.$$

The given Series is Convergent.

By alternating Series test,

$$\left.\begin{array}{ll} (1) & \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} \checkmark \\ (2) & \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \checkmark \end{array}\right\} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} : \text{Convergent}$$

However the Series is not absolutely Convergent:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} : \text{divergent } (p\text{-series with } p = \tfrac{1}{2})$$

8. Write down the degree 4 Taylor polynomial centered at 0:

$$p_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k$$

for $f(x) = 1 + e^{-x}$.

$f(0) = 2$

$f'(x) = -e^{-x}$ $\quad f'(0) = -1$

$f''(x) = e^{-x}$ $\quad f''(0) = 1$

$f'''(x) = -e^{-x}$ $\quad f'''(0) = -1$

$f''''(x) = e^{-x}$ $\quad f''''(0) = 1$

$$P_4(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4$$

$$= 2 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4$$

9. Find the interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n}}.$$

(Verify and clearly mention whether your final answer is a(n) open, half-open, or closed interval!)

$$L = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}/\sqrt{n+1}}{(x-2)^n/\sqrt{n}} \right| = \left(\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \right) |x-2| < 1$$

So the power series converges when

$1 < x < 3$,

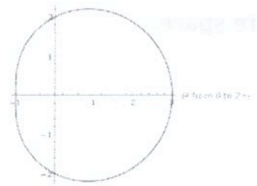
while $x = 1$ and $x = 3$: inconclusive.

When $x=1$, $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n}} = \sum_{n=1} \frac{(-1)^n}{\sqrt{n}}$: Convergent by alternating series test (see #7 solution)

when $x=3$ $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$: divergent. (p-series with $p = \frac{1}{2}$)

Hence the interval of convergence is

$$1 \leq x < 3.$$



10. (1) (3 points) Let C be a circle of radius 2 centered at $(0, 2)$. Write the equation of C in the polar coodinate.

$$x^2 + (y-2)^2 = 4.$$

$$\Leftrightarrow r^2\cos^2\theta + (r\sin\theta - 2)^2 = 4$$

$$\Leftrightarrow r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta + 4 = 4.$$

$$\Leftrightarrow r^2 - 4r\sin\theta = 0.$$

$$\Leftrightarrow r = 4\sin\theta$$

(2) (10 points) Calculate the enclosed area by the limaçon $r = 2 + \cos\theta$ depicted as above.

$$A = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} 4 + 4\cos\theta + \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}} \, d\theta$$

$$= \frac{1}{2}\left[4\theta + 4\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= 4\pi + \pi = \underline{\underline{5\pi}} \,/\!/$$

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