Exam II
MATH 155 Section 08
November 5th, 2015. 7:35PM–9:25PM

• Minor calculation mistake −1 pt.

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Instructions: Please clearly write your name above. This exam is closed-book and closed-note. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Write all details explicitly. Answers without justifications and/or calculation steps may receive no score. Hand-in this exam sheets and other sheets which contain your work to be graded.

Total 100 points. 10 points each unless specified otherwise.

1. Evaluate the following integral:

$$\int \cos^3 x dx$$

$$\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x dx = \int (1-\sin^2 x)\cos x dx = ☆$$

Correct setup +3

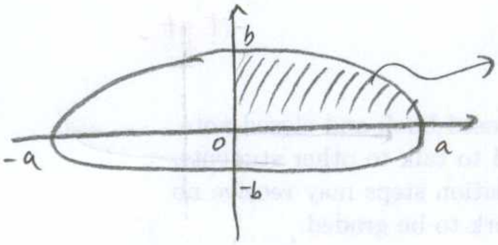
$$\left[\text{put } \begin{array}{l} t = \sin x. \\ dt = \cos x \, dx. \end{array} \right]$$

Correct substitution +2

$$☆ = \int (1-t^2) dt = t - \frac{1}{3}t^3 + C = \sin x - \frac{1}{3}\sin^3 x + C.$$

Correct answer +5.

2. Prove that the area of an ellipse, whose equation is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is $ab\pi$.



$$A = \int_0^a y\,dx.$$

Correct Setup +3.

Calculate y: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff b^2x^2 + a^2y^2 = a^2b^2.$

$$a^2y^2 = a^2b^2 - b^2x^2$$

$$y^2 = b^2 - \frac{b^2}{a^2}x^2.$$

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$

$$A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

Put $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$ (x: $0 \to a$, θ: $0 \to \frac{\pi}{2}$)

Correct Substitution +3

$$= \int_0^{\frac{\pi}{2}} \frac{b}{a}\sqrt{a^2 - a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$$

$$= ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta = ab\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}ab.$$

So the area of the given ellipse is πab. ✓

Correct answer +4.

3. Evaluate

$$\int \frac{dx}{(1+x^2)^2}.$$

Put $x = \tan\theta$.
$dx = \sec^2\theta\,d\theta$

Correct Substitution +6

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\,d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C.$$

Correct answer +4

4. Evaluate
$$\int \frac{dx}{x^2 - 5x + 6}.$$

$$\int \frac{dx}{x^2-5x+6} = \int \frac{1}{(x-2)(x-3)} dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2} \right) dx = \underline{\ln|x-3| - \ln|x-2| + C}$$

Correct partial fractions +5.

· Correct answer +5

5. Find the constant k that satisfies the following equation:
$$\int_{-\infty}^{\infty} \frac{k}{1+4x^2} dx = 1.$$

Recall: $\int \frac{1}{1+4x^2} dx \underset{\uparrow}{=} \int \frac{1}{1+t^2} \cdot \frac{1}{2} dt = \frac{1}{2} \int \frac{1}{1+t^2} dt = \frac{1}{2}\tan^{-1} t + C$

put $t = 2x$

$dt = 2dx$

$= \underline{\frac{1}{2}\tan^{-1} 2x + C}.$

Correct indefinite integral +3

$$\text{LHS} = \underline{\lim_{R\to\infty} \int_{-R}^{R} \frac{k}{1+4x^2} dx} = \lim_{R\to\infty} \frac{k}{2}\tan^{-1} 2x \Big|_{-R}^{R} = \frac{k}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{k\pi}{2}.$$

RHS = 1.

Correct Setup of improper integral +4

From LHS = RHS, $k = \underline{\underline{\frac{2}{\pi}}}$.

Correct answer +3.

6. (5 points each) Let $f(x) = \frac{1}{x^p}$, where $0 < p < \infty$. Discuss the convergence of the definite integral $\int_1^\infty f(x)dx$ in the following cases:

(1) When $0 < p < 1$:

$$\underbrace{\lim_{R\to\infty} \int_1^R \frac{1}{x^p}\,dx}_{} = \lim_{R\to\infty} \frac{1}{1-p} x^{1-p}\Big|_1^R = \lim_{R\to\infty} \frac{R^{1-p}}{1-p} - \frac{1}{1-p}$$

$1-p > 0$

Correct setup of improper integral +3

$$= \infty \quad (\because R^{1-p} \to \infty \text{ as } R \to \infty)$$

Correct answer +2

(2) When $p = 1$:

$$\underbrace{\lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx}_{} = \lim_{R\to\infty} \ln|x|\Big|_1^R = \lim_{R\to\infty} \ln|R| = \infty$$

Correct setup of improper integral +3

Correct answer +2

(3) When $p > 1$:

$$\underbrace{\lim_{R\to\infty} \int_1^R \frac{1}{x^p}\,dx}_{} = \lim_{R\to\infty} \frac{1}{1-p} x^{1-p}\Big|_1^R = \lim_{R\to\infty} \frac{R^{1-p}}{1-p} - \frac{1}{1-p} = \frac{1}{p-1}.$$

$R^{1-p} \to 0$ as $R \to \infty$

Correct setup of improper integral +3

Correct answer +2.

7. Find the value that the following infinite sum converges to:

$$S = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

Partial Sum $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \underline{\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)}$

$$= 1 - \cancel{\frac{1}{2}}$$
$$+ \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$
$$+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$$
$$+ \cdots\cdots$$
$$+ \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 ✓$$

Correct Partial Sum + partial fraction decomposition +5

Correct telescopic sum +4

Correct answer +1.

8. Find the limit of the sequence:

$$a_n = \frac{\sin n}{n^2+1}.$$

No partial Credit. +10 for only Complete answer.

Let $b_n = \frac{-1}{n^2+1}$, $C_n = \frac{1}{n^2+1}$.

$$b_n \le a_n \le C_n$$

$$-\frac{1}{n^2+1} \le \frac{\sin n}{n^2+1} \le \frac{1}{n^2+1}$$

because $-1 \le \sin n \le 1$ for all n.

Since $\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} C_n$,

$\lim_{n\to\infty} a_n = 0$, by the Squeeze theorem.

9. Evaluate the following geometric series: $1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \ldots + \frac{1}{\pi^n} + \ldots$.

First term $a = 1$

Ratio $r = \frac{1}{\pi}$.

Geometric Series $= \frac{a}{1-r} = \frac{1}{1-\frac{1}{\pi}} = \underline{\underline{\frac{\pi}{\pi-1}}}$ ✓

No partial credit

+10 for correct answer with correct details.

10. (5 points) Evaluate $\int \ln x dx$. (Hint: Integration by parts.)

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \ln x - \int dx$$

$$= \underline{x \ln x - x + C.}$$

Correct application of integration by parts +3

Correct answer +5.