Midterm Exam
MATH 25500 Section 01
24th March 2017, 14:10–15:25

Your name:

Minor errors or Calculation mistakes : −2 pts to −1 pt.

Instructions: Please clearly write your name above. You may use your textbook or notes. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Please write all details explicitly. Answers without justifications and/or calculation steps may receive no score.

1. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be C^1-functions. Show that $\nabla(fg) = f\nabla g + g\nabla f$.

$$\nabla(fg) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(fg)\hat{e}_i = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i} g + f\frac{\partial g}{\partial x_i}\right)\hat{e}_i = \left(\sum_{i=1}^{3}\frac{\partial f}{\partial x_i}\right) g\hat{e}_i + f\left(\sum_{i=1}^{3}\frac{\partial g}{\partial x_i}\right)\hat{e}_i = g\nabla f + f\nabla g.$$

where $\hat{e}_1 = \hat{i}$, $\hat{e}_2 = \hat{j}$, and $\hat{e}_3 = \hat{k}$

−5 points if you proved
$\nabla\cdot(fg) = f\nabla\cdot g + g\nabla\cdot f$.

In #2 and #3 of this exam, ~~From here to the end of this exam,~~ $\vec{F}(x,y,z) = (x,y,z)$ is a vector field on $\mathbb{R}^3$.

2. Find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$.

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = \left(\partial_y z - \partial_z y ,\ \partial_z x - \partial_x z ,\ \partial_x y - \partial_y x \right)$$

$$= (0, 0, 0)$$

3. Let $\alpha(t) = (t \sin t, t \cos t, t)$ be a curve in $\mathbb{R}^3$ defined on $[0, 2\pi]$. Calculate $\int_\alpha \vec{F} \cdot d\vec{r}$.

Correct setup of line integral : 5pts

Let $f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$.

Note $\vec{F} = \nabla f = (x, y, z)$.

$$\int_\alpha \vec{F} \cdot d\vec{r} = \int_\alpha \nabla f \cdot d\vec{r} = f(\alpha(2\pi)) - f(\alpha(0))$$

$$= f(0, 2\pi, 2\pi) - f(0,0,0)$$

$$= \frac{4\pi^2}{2} + \frac{4\pi^2}{2} = \underline{4\pi^2}.$$

From here to the end of this exam, $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$(u, v) \mapsto (v \sin u, v \cos u, v)$$

is a parametrization of a surface S, where $D = [0, 2\pi] \times [0, 1]$.

4. Find all points $(u, v) \in D$ such that the surface S is regular at $\Phi(u, v)$.

$$\vec{T}_u = \frac{\partial \Phi}{\partial u} = (v \cos u, -v \sin u, 0)$$

$$\vec{T}_v = \frac{\partial \Phi}{\partial v} = (\sin u, \cos u, 1)$$

Incorrect answer on regularity : -3 pts.

No regularity discussion: -5 pts.

$$\vec{T}_u \times \vec{T}_v = (-v \sin u, -v \cos u, v) \neq \vec{0} \quad \text{iff } v \neq 0.$$

Hence the surface S is regular on $\{(u, v) \in D : u \in [0, 2\pi] \text{ and } v \in (0, 1]\}$.

(i.e. S is regular except at $(0,0,0)$)

5. Find the area of S.

$$\text{Area}(S) = \iint_S dS = \iint_D \|\vec{T}_u \times \vec{T}_v\| \, du\, dv = \int_0^{2\pi}\int_0^1 \sqrt{2}\, v \, dv\, du = \frac{\sqrt{2}}{2} v^2 \Big|_0^1 \cdot 2\pi$$

$$= \sqrt{2}\,\pi.$$

incorrect double integral -2 pts

6. ~~(15 points)~~ Evaluate $\iint_S x dS$.

Let $f(x, y, z) = x$.

$$\iint_S f \, dS = \iint_D f(\Phi(u,v)) \|\vec{T}_u \times \vec{T}_v\| \, du \, dv$$ 2pts

$$= \int_0^1 \int_0^{2\pi} v \sin u \cdot \sqrt{2} v \, du \, dv$$

$$= \int_0^1 \sqrt{2} v^2 \int_0^{2\pi} \sin u \, du \, dv = \int_0^1 \sqrt{2} v^2 \left[-\cos u \Big|_0^{2\pi} \right] dv$$

$$= 0$$

−2pts for incorrect double integral

7. ~~(15 points)~~ Compute the surface integral $\iint_S \vec{G} \cdot d\vec{S}$, where $\vec{G}(x, y, z) = (x, y, z^2)$

$$\iint_S \vec{G} \cdot d\vec{S} = \iint_D \vec{G} \cdot \vec{T}_u \times \vec{T}_v \, du \, dv$$

$$= \iint_D (v \sin u, v \cos u, v^2) \cdot (-v \sin u, -v \cos u, v) \, du \, dv$$ 5pts

$$= \iint_D -v^2 \sin^2 u - v^2 \cos^2 u + v^3 \, du \, dv$$

incorrect double integral: −2pts.

$$= 2\pi \int_0^1 v^3 - v^2 \, dv = 2\pi \left(\frac{1}{4} v^4 - \frac{1}{3} v^3 \right) \Bigg|_0^1$$

$$= 2\pi \left(\frac{1}{4} - \frac{1}{3} \right) = -\frac{1}{6} \pi$$

and the mean curvature

8. (20 points) Calculate the Gauss curvature at each point $p \in S - \{(0,0,0)\}$.

$\vec{T}_u = \frac{\partial \Phi}{\partial u} = (v\cos u, -v\sin u, 0)$

$\vec{T}_v = \frac{\partial \Phi}{\partial v} = (\sin u, \cos u, 1)$

$E = \left\|\frac{\partial \Phi}{\partial u}\right\|^2 = v^2$ +2

$F = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} = 0$ +2

$G = \left\|\frac{\partial \Phi}{\partial v}\right\|^2 = 2$ +2

$W = EG - F^2 = 2v^2$

$\vec{N} = \frac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|} = \left(-\frac{\sqrt{2}}{2}\sin u, -\frac{\sqrt{2}}{2}\cos u, \frac{\sqrt{2}}{2}\right)$ +2

$\vec{T}_{uu} = (-v\sin u, -v\cos u, 0)$ +2

$\vec{T}_{uv} = (\cos u, -\sin u, 0)$ +2

$\vec{T}_{vv} = (0, 0, 0)$ +2.

$L = \vec{N} \cdot \vec{T}_{uu} = \frac{\sqrt{2}}{2}v$ +2

$M = \vec{N} \cdot \vec{T}_{uv} = 0$ +2

$N = \vec{N} \cdot \vec{T}_{vv} = 0$ +2

$$K(p) = \frac{LN - M^2}{W} = \frac{0 - 0}{2v^2} = 0$$

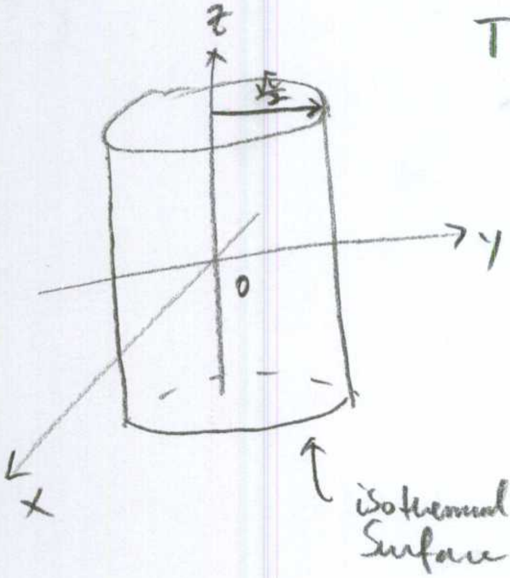
You can also argue that two principal curvatures at $\vec{p} = (v\sin u, v\cos u, v)$ are 1 and 0

Hence $k(\vec{p}) = 1 \cdot 0 = 0$.

$$H(p) = \frac{GL + En - 2FM}{2W} = \frac{\sqrt{2}v + 0 - 0}{4v^2} = \frac{\sqrt{2}}{4v}$$

9. The temperature distribution in $\mathbb{R}^3$ is given by $T(x, y, z) = 4x^2 + 4y^2$. Compute the heat flux across the surface $x^2 + y^2 = 1$ and $0 \leq z \leq 1$. Here the heat flux $\vec{F}$ is defined by $\vec{F} := -\nabla T$.

$T = C = 4x^2 + 4y^2$



$$\vec{F} = -\nabla T = (-8x, -8y, 0)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = -8 \iint_S dS = -8 \cdot 2\pi \cdot 1 = -16\pi$$

$\Phi : (\theta, z) \mapsto (\cos\theta, \sin\theta, z)$

defined on $[0, 2\pi] \times [0, 1]$

$\vec{T}_\theta = (-\sin\theta, \cos\theta, 0)$

$\vec{T}_z = (0, 0, 1)$

$\vec{T}_\theta \times \vec{T}_z = (\cos\theta, \sin\theta, 0)$

unit normal $= \vec{n}$

$\vec{F}(\Phi(\theta, z)) = (-8\cos\theta, -8\sin\theta, 0)$

$\vec{F}(\Phi(\theta, z)) \cdot \vec{n} = -8$