Final Examination
MATH 25500 Section 01
19th May 2017, 14:10–15:25

Your name:

Instructions: Please clearly write your name above. This exam is closed-book and closed-notes. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Please write all details explicitly. Answers without justifications and/or calculation steps may receive no score. 10 points each unless stated otherwise.

1. (5 points) Evaluate the integral

$$\int_C (7x^3 \cos x - 2y^3)dx + (2x^3 + y^3 e^{-y})dy$$

where C is the unit circle in $\mathbb{R}^2$.

2. Use Green's theorem to calculate the area of the ellipse defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3.(5 points) Compute the line integral

$$\int_C x^{2017} dx + y^{2017} dy$$

along the unit circle C in $\mathbb{R}^2$.

4. If C is a closed curve that is the boundary of a surface S, and f and g are C^2 functions, show that

$$\int_C (f\nabla g + g\nabla f) \cdot d\overrightarrow{r} = 0.$$

5. Suppose a C^1-vector field $\overrightarrow{F}$ is defined on any region D in $\mathbb{R}^2$ that Green's theorem applies. Give an example of a vector field $\overrightarrow{F}$, a region D, and a curve C in D such that $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is nonvanishing.

6. Evaluate $\iint_S \overrightarrow{F} \cdot d\overrightarrow{S}$, where $\overrightarrow{F} = 3xy^2\overrightarrow{i} + 3x^2y\overrightarrow{j} + z^3\overrightarrow{k}$ and S is the surface of the unit sphere in $\mathbb{R}^3$.

7. Suppose an electric charge Q is placed at $(0,0,0)$. At $\overrightarrow{r} = (x,y,z)$, the electric field $\overrightarrow{E}$ is given by

$$\overrightarrow{E}(\overrightarrow{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{\|r\|^3}\overrightarrow{r}.$$

Calculate $\iint_S \overrightarrow{E} \cdot d\overrightarrow{S}$, where S is the unit sphere. If necessary, you can use one of Maxwell's equations $\nabla \cdot \overrightarrow{E} = \frac{\rho}{\epsilon_0}$, where ρ is the electric charge density function and ϵ_0 is a constant.

8. Let φ, ψ, and θ be the following differential forms in $\mathbb{R}^3$.

$$\varphi = xdx - ydy$$

$$\psi = zdx \wedge dy + xdy \wedge dz$$

$$\theta = zdy.$$

Compute $\varphi \wedge \psi$ and $d\psi$.

9. (10 points each) (1) Let ξ be a differential 1-form satisfying the differential equation $d\xi = 0$. The operator $d_\xi(-) := d(-) + \xi\wedge(-)$ acts on an arbitrary differential form ω by $\omega \mapsto d\omega + \xi\wedge\omega$. Show that $d_\xi^2 = 0$.

(2) Let d_ξ be as above. Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable map. Show that $f^* \circ d_\xi = d_{f^*\xi} \circ f^*$.

Such a "twisted" de Rham complex defines a bigraded "twisted" de Rham cohomology called the *Morse-Novikov cohomology*.

10. Given a k-form ω in $\mathbb{R}^n$ we will define an $(n-k)$-form $*\omega$ by setting

$$*(dx_{i_1}\wedge\cdots\wedge dx_{i_k}) = (-1)^{\sigma}(dx_{j_1}\wedge\cdots\wedge dx_{j_{n-k}})$$

and extending it linearly, where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$, $(i_1, \cdots, i_k, j_1, \cdots, j_{n-k})$ is a permutation of $(1, 2, \cdots, n)$ and σ is 0 or 1 according to the permutation is even or odd, respectively. Show that:

$$**\omega = (-1)^{k(n-k)}\omega.$$

Please use this space if you need more space.