Final Examination
MATH 25500 Section 01
19th May 2017, 14:10–15:25

Your name:

Instructions: Please clearly write your name above. This exam is closed-book and closed-notes. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Please write all details explicitly. Answers without justifications and/or calculation steps may receive no score. 10 points each unless stated otherwise.

1. (5 points) Evaluate the integral

$$\int_C (7x^3 \cos x - 2y^3)dx + (2x^3 + y^3 e^{-y})dy$$

where C is the unit circle in $\mathbb{R}^2$.

let $P(x,y) = 7x^3 \cos x - 2y^3$
$Q(x,y) = 2x^3 + y^3 e^{-y}$

$\frac{\partial Q}{\partial x} = 6x^2 \quad , \quad \frac{\partial P}{\partial y} = -6y^2$

By Green's theorem, the given integral is

$$\iint_{D=\{(x,y):\ x^2+y^2\le 1\}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dxdy = \iint_D 6(x^2+y^2)\, dxdy = 6\int_0^1\int_0^{2\pi} r^3\, d\theta dr$$

$$= 6\int_0^1 2\pi \cdot r^3 dr = 12\pi\left[\tfrac{1}{4}r^4\right]_0^1 = \underline{\underline{3\pi}}.$$

2. Use Green's theorem to calculate the area of the ellipse defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Parametrization $x = a\cos t$ $\quad t \in [0, 2\pi]$.
$y = b \sin t$

$$\text{Area} = \iint_{D=\{(x,y):\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}} 1 \cdot dA = \frac{1}{2}\int_{C \text{ given ellipse}} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos t \cdot b\cos t - b\sin t \cdot (-a\sin t)\, dt$$

$$\frac{dx}{dt} = -a\sin t$$
$$\frac{dy}{dt} = b\cos t$$

$$= \frac{ab}{2}\int_0^{2\pi} 1\, dt = ab\pi. \checkmark$$

3.(5 points) Compute the line integral

$$\int_C x^{2017}dx + y^{2017}dy$$

along the unit circle C in $\mathbb{R}^2$.

Let $\vec{F}(x,y,z) = (x^{2017}, y^{2017}, 0)$.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^{2017} & y^{2017} & 0 \end{vmatrix} = (0, 0, 0).$$

By Stokes' theorem, the given integral is

$$\iint_{D=\{(x,y,z):\ x^2+y^2\le 1,\ z=0\}} (\nabla \times \vec{F}) \cdot d\vec{S} = 0. \checkmark$$

4. If C is a closed curve that is the boundary of a surface S, and f and g are C^2 functions, show that

$$\int_C (f\nabla g + g\nabla f)\cdot d\vec{r} = 0.$$

Note that $\nabla\times(f\nabla g) = \nabla f\times\nabla g + f\,\nabla\times(\nabla g)$.

So $\displaystyle\int_C f\nabla g\cdot d\vec{r} = \iint_S (\nabla f\times\nabla g)\cdot d\vec{S}$ $\quad\cdots$ ①

Similarly $\displaystyle\int_C g\nabla f\cdot d\vec{r} = \iint_S (\nabla g\times\nabla f)\cdot d\vec{S} = -\iint_S (\nabla f\times\nabla g)\cdot d\vec{S}$ $\quad\cdots$ ②

$\times$ is skew-symmetric.

Adding ① and ② we get the result.

5. Suppose a C^1-vector field $\vec{F}$ is defined on any region D in $\mathbb{R}^2$ that Green's theorem applies. Give an example of a vector field $\vec{F}$, a region D, and a curve C in D such that $\int_C \vec{F}\cdot d\vec{r}$ is nonvanishing.

Let $D := \{(x,y)\in\mathbb{R}^2 : \frac{1}{4}\le x^2+y^2\le 4\}$.

Take $C := \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$ parametrized by $r(t) = (\cos t, \sin t)$ on $[0,2\pi]$.

Let $\vec{F} = (y, -x)$. Then $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin t(-\sin t) - \cos t\cdot\cos t\,dt$

$\displaystyle = -\int_0^{2\pi} 1\,dt = -2\pi.$

6. Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = 3xy^2\vec{i} + 3x^2y\vec{j} + z^3\vec{k}$ and S is the surface of the unit sphere in $\mathbb{R}^3$. (Hint: $dV = r^2 \sin\phi \, dr \, d\theta \, d\phi$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$).

$$\nabla \cdot \vec{F} = 3y^2 + 3x^2 + 3z^2$$

By Gauss' divergence theorem,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V 3(x^2+y^2+z^2)\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 3r^2 \, r^2 \sin\phi \, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{3}{5} r^5 \Big|_0^1 \sin\phi \, d\theta\, d\phi = \frac{6\pi}{5}\int_0^{\pi} \sin\phi \, d\phi = -\frac{6\pi}{5}\cos\phi \Big|_0^{\pi}$$

$$= \underline{\underline{\frac{12\pi}{5}}}$$

7. Suppose an electric charge Q is placed at $(0,0,0)$. At $\vec{r} = (x,y,z)$, the electric field $\vec{E}$ is given by

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{\|r\|^3}\vec{r}.$$

Calculate $\iint_S \vec{E} \cdot d\vec{S}$, where S is the unit sphere. If necessary, you can use one of Maxwell's equations $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$, where ρ is the electric charge density function and ϵ_0 is a constant.

$$\iint_S \vec{E} \cdot d\vec{S} \underset{\substack{\uparrow \\ \text{Gauss' divergence} \\ \text{theorem}}}{=} \iiint_V \nabla \cdot \vec{E}\, dV = \frac{1}{\varepsilon_0}\iiint \rho \, dV = \frac{Q}{\varepsilon}.$$ ✓

8. Let φ, ψ, and θ be the following differential forms in $\mathbb{R}^3$.

$$\varphi = xdx - ydy$$

$$\psi = zdx \wedge dy + xdy \wedge dz$$

$$\theta = zdy.$$

Compute $\varphi \wedge \psi$ and $d\psi$.

$$\varphi \wedge \psi = (xdx - ydy) \wedge (zdx \wedge dy + xdy \wedge dz)$$

$$= 0 - 0 + x^2 dx \wedge dy \wedge dz - 0 = \underline{\underline{x^2 dx \wedge dy \wedge dz}}$$

$$d\psi = dz \wedge dx \wedge dy + dx \wedge dy \wedge dz = \underline{\underline{2\, dx \wedge dy \wedge dz}}$$

9. (10 points each) (1) Let ξ be a differential 1-form satisfying the differential equation $d\xi = 0$. The operator $d_\xi(-) := d(-) + \xi\wedge(-)$ acts on an arbitrary differential form ω by $\omega \mapsto d\omega + \xi\wedge\omega$. Show that $d_\xi^2 = 0$.

(2) Let d_ξ be as above. Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable map. Show that $f^* \circ d_\xi = d_{f^*\xi} \circ f^*$.

Such a "twisted" de Rham complex defines a bigraded "twisted" de Rham cohomology called the *Morse-Novikov cohomology.*

(1) For any differential form ω

$$
\begin{aligned}
d_\xi^2\,\omega &= d_\xi(d\omega + \xi\wedge\omega) = d^2\omega + \xi\wedge d\omega + d(\xi\wedge\omega) + \xi\wedge(\xi\wedge\omega) \\
&= 0 + \xi\wedge d\omega + \cancel{d\xi\wedge\omega}^{\,0} - \xi\wedge d\omega \\
&\qquad + 0 \\
&\qquad (\xi\wedge\xi = 0) \\
&= 0
\end{aligned}
$$

(2) For any differential form ω,

$$
\begin{aligned}
f^*\circ d_\xi\,\omega &= f^* d\omega + f^*(\xi\wedge\omega) \\
&= d f^*\omega + f^*\xi \wedge f^*\omega \\
&= d_{f^*\xi}\circ f^*\omega \quad ✓
\end{aligned}
$$

10. Given a k-form ω in $\mathbb{R}^n$ we will define an $(n-k)$-form $*\omega$ by setting

$$*(dx_{i_1}\wedge\cdots\wedge dx_{i_k}) = (-1)^\sigma(dx_{j_1}\wedge\cdots\wedge dx_{j_{n-k}})$$

and extending it linearly, where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$, $(i_1,\cdots,i_k,j_1,\cdots,j_{n-k})$ is a permutation of $(1,2,\cdots,n)$ and σ is 0 or 1 according to the permutation is even or odd, respectively. Show that:

$$**\omega = (-1)^{k(n-k)}\omega.$$

Let $\omega = \sum_I f_I \, dx_I$ where $I = (i_1, \cdots, i_k)$

$* \omega = \sum_I f_I *(dx_I) = \sum_I f_I (-1)^{\sigma(IJ)} dx_{J(I)}$ where $(IJ) = (i_1, \cdots, i_k, j_1, \cdots, j_{n-k})$

$** \omega = \sum_I f_I (-1)^{\sigma(JI)} (-1)^{\sigma(IJ)} dx_I \underset{\uparrow}{=} (-1)^{k(n-k)} \omega.$ ✓

$(-1)^{\sigma(JI)} = (-1)^{\sigma(IJ)} (-1)^{k(n-k)}.$

Please use this space if you need more space.