Aunt Debi
Math 255
5/12/17

Homework #10
section 8.4

2/2 Excellent!

5) $F=(x-y, y-z, z-x)$ out of the unit sphere

$\iiint_W (\text{div} F)\,dV$

$\text{div} F = 1+1+1=3$

$\iiint_W 3\,dV = 3\iiint_W dV = \frac{3\cdot 4\pi\cdot R^3}{3}$ where $R=1$, the radius of unit sphere.

$\rightarrow \frac{3\cdot 4\pi \cdot 1}{3} = \boxed{4\pi}$

7) $F=(x,y,z)$ w/ the unit sphere.

$\iint_{\partial W} F\cdot dS = \iiint_W (\nabla\cdot F)\,dV$

$= \int_0^1\int_0^1\int_0^1 (1+1+1)\,dV$

$= (3)\cdot(1)\cdot(1)\cdot(1) = \boxed{3}$

9b) $F=(y, z, xz)$, $W: x^2+y^2 \le z \le 1,\ x \ge 0$

$\iint_{\partial W} F\cdot dS = \iiint_W (\text{div} F)\,dV = \iiint_W 0+0+x\,dV$

$(x,y,z)\rightarrow(r\cos\theta, r\sin\theta, z)$ (cylindical)

$\iiint_W x\,dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\int_{r^2}^1 r\cos\theta\cdot r\,dz\,dr\,d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\int_{r^2}^1 r^2\cos\theta\,dz\,dr\,d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 zr^2\cos\theta\Big|_{r^2}^1 dr\,d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 \cos\theta(r^2-r^4)\,dr\,d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\left(\frac{r^3}{3}-\frac{r^5}{5}\right)\Big|_0^1 d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\left(\frac{2}{15}\right) d\theta$

$= \frac{2}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,d\theta$

$= \frac{2}{15}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{2}{15}(1-(-1)) = \frac{2}{15}\cdot 2 = \boxed{\frac{4}{15}}$

12) $F = (3xy^2, 3x^2y, z^3)$, S is the surface of unit sphere

$$\iint_{\partial W} F \cdot dS = \iiint_{\partial W} \operatorname{div} F \, dS = \iiint_{\partial W} 3y^2 + 3x^2 + 3z^2 \, dV$$

$$= 3 \iiint_{\partial W} x^2 + y^2 + z^2 \, dV$$

$(x, y, z) \to (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$ (spherical)

$$3 \iiint_{\partial W} x^2 + y^2 + z^2 \, dW =$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 (\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 (\sin^2\phi \cos^2\theta + \sin^2\phi \sin^2\theta + \cos^2\phi) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 [\sin^2\phi \, (\overbrace{\cos^2\theta + \sin^2\theta}^{=1}) + \cos^2\phi] \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 (\sin^2\phi + \cos^2\phi) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi} \frac{\rho^5}{5} \sin\phi \Big|_0^1 \, d\phi \, d\theta$$

$$= \frac{3}{5} \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{3}{5} \int_0^{2\pi} -\cos\phi \Big|_0^{\pi} \, d\theta$$

$$= \frac{3}{5} \int_0^{2\pi} -(-1) - (-1) \, d\theta$$

$$= \frac{6}{5} \int_0^{2\pi} d\theta = \frac{6}{5} \cdot 2\pi = \boxed{\frac{12\pi}{5}}$$

16) $F(x, y, z) = (1, 1, z(x^2+y^2)^2)$, $\partial S: x^2 + y^2 \leq 1, \ 0 \leq z \leq 1$

$$\iint_S F \cdot n \, dS = \iiint_W (\operatorname{div} F) \, dx \, dy \, dz = \iiint_W (x^2+y^2)^2 \, dx \, dy \, dz$$

$(x, y, z) \longrightarrow (r\cos\theta, r\sin\theta, z)$ (cylindrical)

$$\iiint_W (x^2+y^2)^2 \, dx \, dy \, dz = \int_0^1 \int_0^{2\pi} \int_0^1 ((r\cos\theta)^2 + (r\sin\theta)^2)^2 \cdot r \, dr \, d\theta \, dz$$

$$= \int_0^1 \int_0^{2\pi} \int_0^1 (r^2)^2 \cdot r \, dr \, d\theta \, dz$$

$$= \int_0^1 \int_0^{2\pi} \int_0^1 r^5 \, dr \, d\theta \, dz$$

$$= \int_0^1 \int_0^{2\pi} \frac{r^6}{6} \Big|_0^1 \, d\theta \, dz$$

$$= \int_0^1 \int_0^{2\pi} \frac{1}{6} \, d\theta \, dz$$

$$= \frac{1}{6} \cdot 2\pi \cdot 1 = \boxed{\frac{\pi}{3}}$$

17) Prove that $\iiint_W (\nabla f)\cdot F\,dxdydz = \iint_{\partial W} fF\cdot n\,ds - \iiint_W f\,\nabla\cdot F\,dxdydz$

Vector identity: $\nabla\cdot(fF) = \nabla f\cdot F + f\nabla\cdot F$

$\rightarrow \nabla f\cdot F = \nabla\cdot(fF) - f\nabla\cdot F$

$$\iiint_W (\nabla f)\cdot F\,dxdydz = \iiint_W \nabla\cdot(fF)\,dxdydz - \iiint_W f\nabla\cdot F\,dxdydz$$

$$\iiint_W \cancel{f}(\nabla\cdot F)\,dxdydz$$

$$\iiint_W (\nabla f)\cdot F\,dxdydz = \iiint_W fF\cdot n\,ds - \iiint_W f\nabla\cdot F\,dxdydz$$

19) Show that $\iiint_W \left(\frac{1}{r^2}\right) dxdydz = \iint_{\partial W}\left(\frac{\vec{r}\cdot\vec{n}}{r^2}\right) ds$, where $r = (x, y, z)$.

If $F = \frac{\vec{r}}{r^2}$, then $\nabla\cdot F = \frac{1}{r^2}$. If $(0,0,0)\notin\Omega$, then follow Gauss's Theorem. If $(0,0,0)\in\Omega$, compute the integral by deleting a small ball $B_\varepsilon = \{(x,y,z) \mid (x^2+y^2+z^2)^{1/2} < \varepsilon\}$ around the origin and letting $\varepsilon\to 0$:

$$\iiint_\Omega \frac{1}{r^2}dV = \lim_{\varepsilon\to 0}\iiint_{\Omega \setminus B_\varepsilon} \frac{1}{r^2}dV = \lim_{\varepsilon\to 0}\iint_{\partial\left(\frac{\Omega}{B_\varepsilon}\right)} \frac{\vec{r}\cdot\vec{n}}{r^2}ds$$

$$= \lim_{\varepsilon\to 0}\left(\iint_{\partial\Omega}\frac{\vec{r}\cdot\vec{n}}{r^2}ds - \iint_{\partial B_\varepsilon}\frac{\vec{r}\cdot\vec{n}}{r^2}ds\right)$$

$$= \lim_{\varepsilon\to 0}\left(\iint_{\partial\Omega}\frac{\vec{r}\cdot\vec{n}}{r^2}ds - 4\pi\varepsilon\right)$$

$$= \iint_{\partial\Omega}\frac{\vec{r}\cdot\vec{n}}{r^2}ds$$

21) a) $\iint_{\partial W} f\nabla g\cdot n\,ds = \iiint_W (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV$

$\iint_{\partial W} f\nabla g\cdot n\,ds = \iiint_W f\nabla(\nabla g)\,ds = \iiint_W \nabla(f\nabla g)\,ds$

Vector identity: $\operatorname{div}(fF) = f\operatorname{div}F + F\cdot\nabla f$:

$$\iiint_W \nabla(f\nabla g)\,ds = \iiint_W (f\nabla(\nabla g) + \nabla g\cdot\nabla f)\,dV$$

$$= \iiint_W (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV \quad \blacksquare$$

b) $\iint_{\partial W}(f\nabla g - g\nabla f)\cdot n\,ds = \iiint_W (f\nabla^2 g - g\nabla^2 f)\,dV$

$\iint_{\partial W}(f\nabla g - g\nabla f)\cdot n\,ds = \iiint_W \operatorname{div}(f\nabla g - g\nabla f)\,ds$

Vector identity: $\nabla(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$:

$$\iiint_W \operatorname{div}(f\nabla g - g\nabla f)\,ds = \iiint_W (f\nabla^2 g - g\nabla^2 f)\,ds \quad \blacksquare$$

24) Suppose F is tangent to the closed surface $S = \partial W$ of a region W. Prove that $\iiint_W (\operatorname{div} F)\,dV = 0$

Gauss Theorem: $\iiint_W (\operatorname{div} F)\,dV = \iint_{\partial W} F \cdot n\,dS$

since F is tangent to ∂W, $\vec{F} \cdot \vec{n} = 0$

$\therefore \iint_{\partial W} F \cdot n\,dS = \iint_W 0\,dS = 0$

28) $\iint_S F \cdot dS = \iiint_W (\nabla \cdot F)\,dV$

Let $F = \nabla \times F$:

$\iint_S F\,dS = \iint_S (\nabla \times F)\,dS = \iiint_W (\nabla \cdot (\nabla \times F))\,dV$

Vector identity: $\nabla \cdot (\nabla \times F) = 0$

$\iiint_W (\nabla \cdot \nabla \times F)\,dV = \iiint 0\,dV = 0 = \iint_S (\nabla \times F)\,dS$