MATH 255 — WENJIE LIU

Dr. Byungdo Park — 2/2 Excellent! — 5/12/17

Homework #10

Section 8.4 Exercises 5, 7, 9b, 12, 16, 17, 19, 21, 24, 28

5. Use the divergence theorem to calculate the flux of $F=(x-y)i+(y-z)j+(z-x)k$ out of the unit sphere. $x^2+y^2+z^2=1$

$$\iint_S F\cdot n\, ds = \iiint_W (\text{div } F)\, dv$$

where W is the ball bounded by the sphere

$$\nabla\cdot F = \frac{\partial}{\partial x}(x-y)+\frac{\partial}{\partial y}(y-z)+\frac{\partial}{\partial z}(z-x)$$
$$= 1+1+1=3$$

Because a sphere of radius R has volume $4\pi R^3/3$

$$3\iiint_W (\text{div } F)\, dv = 3\iiint dv = \frac{12\pi}{3} = \boxed{4\pi}$$

7. Evaluate $\iint_{\partial W} F\cdot ds$, where $F=xi+yj+zk$ and W is the unit cube (in the first octant). Perform the calculation directly and check by using the divergence theorem.

$$\iint_{\partial W} F\cdot ds = \int_0^1\int_0^1 (xi+yj+zk)\cdot(-k)\,dxdy = -\int_0^1\int_0^1 z\,dxdy = 0$$

since $z=0$ on xy plane

$$\iint_S F\cdot ds = \int_0^1\int_0^1 (xi+yj+zk)\cdot k\, dxdy = \int_0^1\int_0^1 z\,dxdy = 1$$

since $z=1$ on xy plane

$$\iint_S F\cdot ds = \int_0^1\int_0^1 (xi+yj+zk)\cdot j\, dxdy = \int_0^1\int_0^1 y\,dxdy = 1$$ on xz plane

$$\iint_S F\cdot ds = \int_0^1\int_0^1 (xi+yj+zk)\cdot i\, dxdy = \int_0^1\int_0^1 x\,dxdy = 1$$ on yz plane

then $\iint_{\partial W} F\cdot ds = 1+1+1 = \boxed{3}$

using divergence theorem; $\iint_S F\cdot n\, ds = \iiint_W (\text{div } F)\,dv$

$$\nabla\cdot F = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 1+1+1=3$$

$$\iint_S F\cdot ds = \iiint_W (\text{div } F)\,dv = 3\times \text{volume } (V) = \boxed{3}$$

9. Let $F = y\,i + z\,j + xz\,k$. Evaluate $\iint_{\partial W} F \cdot ds$ for each of the following regions W

(b) $x^2 + y^2 \le z \le 1$ and $x \ge 0$

$$\iint_{\partial W} F \cdot ds = \iiint_W (\operatorname{div} F)\, dv$$

$$\nabla \cdot F = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(xz) = x$$

change to cylindrical coordinate:

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad 0 \le r \le 1$ $\quad r^2 \le z \le 1$ $\quad x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad z = z$

$$\int_{-\pi/2}^{\pi/2}\int_0^1\int_{r^2}^1 r\cos\theta \cdot r\, dz\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1\int_{r^2}^1 r^2\cos\theta\, dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos\theta\, z \Big|_{r^2}^{1} dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos\theta - r^4\cos\theta\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta \int_0^1 r^2 - r^4\, dr = \sin\theta \Big|_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2} - \frac{r^5}{5}\right]_0^1$$

$$= 2\left(\frac{2}{15}\right) = \boxed{4/15}$$

12. Evaluate $\iint_S F \cdot ds$, where $F = 3xy^2 i + 3x^2 y j + z^3 k$ and S is the surface of the unit sphere.

$$\iint_S F \cdot ds = \iiint_{\partial S} (\operatorname{div} F)\, dv$$

$$\nabla \cdot F = \frac{\partial}{\partial x}(3xy^2) + \frac{\partial}{\partial y}(3x^2 y) + \frac{\partial}{\partial z}(z^3) = 3y^2 + 3x^2 + 3z^2 = 3(x^2 + y^2 + z^2)$$

change to spherical coordinate

$0 \le \theta \le 2\pi$ $\quad 0 \le \phi \le \pi$ $\quad 0 \le \rho \le 1$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^2(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^4 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 3\sin\phi \frac{\rho^5}{5}\Big|_0^1 d\phi\, d\theta = \frac{3}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta$$

$$= -\frac{3}{5}\int_0^{2\pi} \cos\phi \Big|_0^{\pi} d\theta = -\frac{3}{5}\int_0^{2\pi} -1-1\, d\theta$$

$$= -\frac{3}{5}(-2\theta)\Big|_0^{2\pi} = \boxed{\frac{12\pi}{5}}$$

16. Evaluate the surface integral $\iint_{\partial S} F\cdot n\,dA$, where $F(x,y,z) = i + j + z(x^2+y^2)^2 k$ and ∂S is the surface of the cylinder $x^2+y^2 \le 1,\ 0 \le z \le 1$

$$\iint_S F\cdot n\,ds = \iiint_W (\operatorname{div} F)\,dV$$

$$\nabla\cdot F = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}\left(z(x^2+y^2)^2\right) = (x^2+y^2)^2$$

change to cylindrical coordinates

$x = r\cos\theta \quad y = r\sin\theta \quad z = z \qquad 0\le r\le 1 \quad 0 \le z \le 1 \quad 0 \le \theta \le 2\pi$

$$\int_0^1\int_0^{2\pi}\int_0^1 (r^2)^2\, r\,dr\,d\theta\,dz = \int_0^1\int_0^{2\pi}\int_0^1 r^5\,dr\,d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi} \frac{r^6}{6}\Big|_0^1 d\theta\,dz = \frac{1}{6}\int_0^1 \theta\Big|_0^{2\pi} dz = \frac{\pi}{3}\Big[z\Big]_0^1 = \boxed{\frac{\pi}{3}}$$

17. prove that

$$\iiint_W (\nabla f)\cdot F\,dx\,dy\,dz = \iint_{\partial W} fF\cdot n\,ds - \iiint_W f\,\nabla\cdot F\,dx\,dy\,dz$$

since $\nabla\cdot(fF) = \nabla f\cdot F + f\,\nabla\cdot F$ and the Gauss' divergence theorem $\iint_S F\cdot n\,ds = \iiint_W (\operatorname{div} F)\,dV$

$$\iiint_W \nabla\cdot(fF)\,dx\,dy\,dz = \iiint_W \nabla f\cdot F\,dx\,dy\,dz + \iiint_W f\,\nabla\cdot F\,dx\,dy\,dz$$

$$\Rightarrow \iiint_W \nabla f\cdot F\,dx\,dy\,dz = \iiint_W \nabla\cdot(fF)\,dx\,dy\,dz - \iiint_W f\,\nabla\cdot F\,dx\,dy\,dz$$

$$= \iiint_W \nabla f\cdot F\,dx\,dy\,dz = \iint_{\partial W} fF\cdot n\,ds - \iiint_W f\,\nabla\cdot F\,dx\,dy\,dz$$

19. Show that $\iiint_W (1/r^2)\,dx\,dy\,dz = \iint_{\partial W} (r\cdot n/r^2)\,ds$, where $r = xi + yj + zk$

If $F = \frac{r}{r^2}$, then $\operatorname{div} F = 1/r^2$. If $(0,0,0)\notin W$, the result follows from Gauss' theorem. If $(0,0,0)\in W$, we compute the integral by deleting a small ball $B_\varepsilon = \{(x,y,z) \setminus (x^2+y^2+z^2)^{1/2} < \varepsilon\}$ around the origin and then letting $\varepsilon \to 0$:

$$\iiint_W \frac{1}{r^2}\,dV = \lim_{\varepsilon\to 0}\iiint_{W/B_\varepsilon} \frac{1}{r^2}\,dV = \lim_{\varepsilon\to 0}\iint_{\partial(W/B_\varepsilon)} \frac{r\cdot n}{r^2}\,ds$$

$$= \lim_{\varepsilon\to 0}\left(\iint_{\partial W} \frac{r\cdot n}{r^2}\,ds - \iint_{\partial B_\varepsilon} \frac{r\cdot n}{r^2}\,ds\right)$$

$$= \lim_{\varepsilon\to 0}\left(\iint_{\partial W} \frac{r\cdot n}{r^2}\,ds - 4\pi\varepsilon\right)$$

$$= \iint_{\partial W} \frac{r\cdot n}{r^2}\,ds$$

21. Prove Green's identities

(a) $\iint_{\partial W} f\nabla g \cdot n\, ds = \iiint_W (f\nabla^2 g + \nabla f \cdot \nabla g)\, dv$ and

(b) $\iint_{\partial W} (f\nabla g - g\nabla f)\cdot n\, ds = \iiint_W (f\nabla^2 g - g\nabla^2 f)\, dv$.

proof (a) By Stoke's Theorem $F = f\nabla g$

$$\int_D \nabla\cdot(f\nabla g)\, dv = \int_{\partial D} f\nabla g \cdot n\, ds$$

From the vector identity $\nabla\cdot(fF) = \nabla f \cdot F + f\nabla\cdot F$

then $\int_{\partial D} f\nabla g \cdot n\, ds = \int_D \nabla f \cdot \nabla g\, dv + \int_D f\nabla^2 g\, dv$

which equal $\iint_{\partial W} f\nabla g \cdot n\, ds = \iiint_W (f\nabla^2 g + \nabla f\cdot\nabla g)\, dv$

proof (b) reversing the roles of f and g in part (a),

$$\iint_{\partial W} g\nabla f \cdot n\, ds = \iiint_W (g\nabla^2 f + \nabla g \cdot \nabla f)\, dv$$

then subtracting last equation from part (a).

$$\iiint_W (f\nabla^2 g - g\nabla^2 f)\, dv = \iint_{\partial W} (f\nabla g - g\nabla f)\cdot n\, ds$$

24. Suppose F is tangent to the closed surface $S = \partial W$ of a region W. Prove that $\iiint_W (\operatorname{div} F)\, dv = 0$

By Gauss divergence theorem, $\iiint_W (\operatorname{div} F)\, dv$ is equal to $\iint_{\partial W} F\cdot n\, ds$. and because F is tangent to the closed surface $S = \partial W$, $F\cdot n = 0$. Thus $\iiint_W (\operatorname{div} F)\, dv = 0$

28. Let S be a closed surface. Use Gauss' theorem to show that if F is a C^2 vector field, then we have $\iint_S (\nabla\times F)\cdot ds = 0$

since from the vector identity $\operatorname{div}(\operatorname{curl} F) = 0$, then

$$\iint_S (\nabla\times F)\cdot ds = \iiint_S \nabla\cdot(\nabla\times F)\, dv = 0$$