Michele Kholodova

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4.3

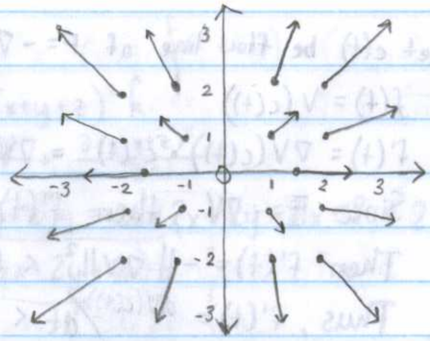
(7) $F(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$F(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$F(1,-1) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$F(2,2) = \left(\frac{2}{\sqrt{8}}, \frac{2}{\sqrt{8}}\right)$

$F(2,1) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

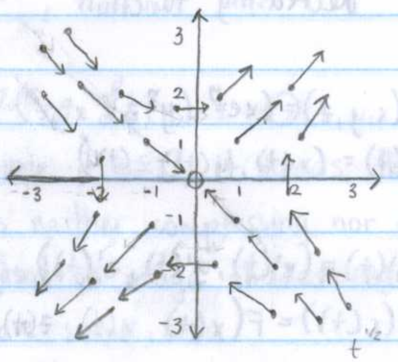


(8) $F(x,y) = \left(\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right)$

$F(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$F(1,-1) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$F(-3,3) = \left(\frac{3}{\sqrt{18}}, -\frac{3}{\sqrt{18}}\right)$



(16) $c(t) = (t^2, 2t-1, \sqrt{t}),\ t > 0 \Rightarrow c'(t) = (2t, 2, \tfrac{1}{2}t^{-1/2}) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right)$

$F(x,y,z) = (y+1, 2, \tfrac{1}{2z})$ $\quad F(c(t)) = F(t^2, 2t-1, \sqrt{t}) = \left(2t, 2, \tfrac{1}{2\sqrt{t}}\right)$

$y+1 = 2t-1+1 = 2t$

*Thus, $F(c(t)) = c'(t)$

(17) $c(t) = (\sin t, \cos t, e^t) \Rightarrow c'(t) = (\cos t, -\sin t, e^t)$ $=$

$F(x,y,z) = (y, -x, z)$ $\quad F(c(t)) = F(\sin t, \cos t, e^t) = (\cos t, -\sin t, e^t)$

*Thus, $F(c(t)) = c'(t)$

(21) a) $F(x,y,z) = (yz, xz, xy)$ $\quad f: \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$

$F = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ $\quad \frac{\partial f}{\partial x} = yz \quad \frac{\partial f}{\partial y} = xz \quad \frac{\partial f}{\partial z} = xy$

$\boxed{f(x,y,z) = xyz}$ $\quad \int yz\,dx \quad \int xz\,dy \quad \int xy\,dz$

b) $F(x,y,z) = (x,y,z)$ $\quad f: \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$

$\frac{\partial f}{\partial x} = x \quad \frac{\partial f}{\partial y} = y \quad \frac{\partial f}{\partial z} = z$

$f(x) = x\,dx \quad f(y) = y\,dy \quad f(z) = z\,dz$

$f(x) = \frac{x^2}{2} \quad f(y) = \frac{y^2}{2} \quad f(z) = \frac{z^2}{2}$

$\boxed{f(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}}$

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(24) Let $c(t)$ be flow line of $F = -\nabla V$. Prove $V(c(t))$ is decreasing function of t.

$f(t) = V(c(t))$ $\quad * \, c'(t) = F(c(t))$

$f'(t) = \nabla V(c(t)) \cdot c'(t) = \nabla V(c(t)) \cdot F(c(t))$

Since $F = -\nabla V$, then $f'(t) = \nabla V \cdot -\nabla V$

Then $f'(t) = -\|\nabla V\|^2 < 0$

Thus, $f'(t) = dV(c(t))/dt < 0$ and $V(c(t))$ is a decreasing function

(27) $F(x,y,z) = (xe^y, y^2z^2, xyz)$

$c(t) = (x(t), y(t), z(t))$ is a flow line for F ~ $c'(t) = F(c(t))$

$c'(t) = (x'(t), y'(t), z'(t))$

$F(c(t)) = F(x(t), y(t), z(t)) = (x(t)e^{y(t)}, y(t)^2 z(t)^2, x(t)y(t)z(t))$

$$\boxed{\begin{aligned} &c'(t) = F(c(t)) \\ &\quad \hookrightarrow x'(t) = x(t)e^{y(t)} \\ &\quad\quad y'(t) = y(t)^2 z(t)^2 \\ &\quad\quad z'(t) = x(t)y(t)z(t) \end{aligned}}$$

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4.4

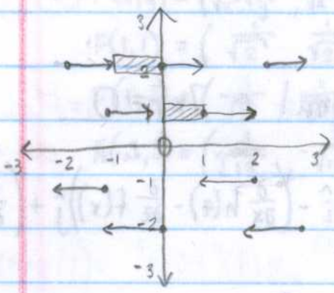
(4) $V(x,y,z) = x^2\hat{i} + (x+y)^2\hat{j} + (x+y+z)^2\hat{k}$

$$\operatorname{div} V = \nabla\cdot V = \frac{\partial x^2}{x} + \frac{\partial (x+y)^2}{y} + \frac{\partial (x+y+z)^2}{z}$$

$$= 2x + 2(x+y) + 2(x+y+z) = 2x + 2x + 2y + 2x + 2y + 2z$$

$$= 6x + 4y + 2z = \boxed{2(3x+2y+z)}$$

(7) $F(x,y) = y\hat{i}$



$\nabla\cdot F = \frac{\partial}{\partial x}(y) = 0$

Since it is 0, it means the fluid is neither compressing nor expanding. Based on image, shaded regions have same area.

(14) $F(x,y,z) = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\hat{i} - \left(\frac{\partial}{\partial x}xy - \frac{\partial}{\partial z}yz\right)\hat{j} + \left(\frac{\partial}{\partial x}xz - \frac{\partial}{\partial y}yz\right)\hat{k}$$

$$= (x-x)\hat{i} - (y-y)\hat{j} + (z-z)\hat{k} = (0,0,0) = 0$$

(17) $F(x,y) = \sin x\,\hat{i} + \cos x\,\hat{j}$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & \cos x & 0 \end{vmatrix} = 0\hat{i} - 0\hat{j} + (-\sin x - 0)\hat{k} = \boxed{(-\sin x)\hat{k}}$$

(21) $F(x,y,z) = (x^2, x^2y, z+zx)$

a) $$\nabla\cdot(\nabla\times\vec{F}) = \nabla\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & x^2y & z+zx \end{vmatrix} = \nabla\cdot\left[(0-0)\hat{i} - (z-0)\hat{j} + (2xy-0)\hat{k}\right]$$

$$\nabla\cdot(0, -z, 2xy) = 0+0+0 = 0 \checkmark$$

$\partial/\partial x \quad \partial/\partial y \quad \partial/\partial z$

b) No, because let there be a f such that $\vec{F} = \nabla\vec{f}$. The curl of a gradient is $\vec{0}$ (ex: $\nabla\times\nabla f = 0$) which implies that $\nabla\times\vec{F} = 0$. But as we saw from (a) $\nabla\times\vec{F} = -z\hat{j} + 2xy\hat{k}$ which does not equal to 0.

(23) $F(x,y,z) = \left(e^{xz},\ \sin(xy),\ x^5y^3z^2\right)$

a) $\nabla \cdot F = \frac{\partial}{\partial x} e^{xz} + \frac{\partial}{\partial y}\sin(xy) + \frac{\partial}{\partial z} x^5y^3z^2$

$$\boxed{= z e^{xz} + x\cos(xy) + 2x^5y^3z}$$

b) $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{xz} & \sin(xy) & x^5y^3z^2 \end{vmatrix} = \left(3y^2x^5z^2 - 0\right)\hat{i} - \left(5x^4y^3z^2 - xe^{xz}\right)\hat{j} + \left(y\cos(xy) - 0\right)\hat{k}$

$$\boxed{= \left(3x^5y^2z^2\right)\hat{i} + \left(xe^{xz} - 5x^4y^3z^2\right)\hat{j} + \left(y\cos(xy)\right)\hat{k}}$$

(26) $f, g, h : \mathbb{R} \to \mathbb{R}$

$F(x,y,z) = \left(f(x), g(y), h(z)\right)$ is irrotational ~ $\nabla \times F = 0$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = \left(\frac{\partial}{\partial y}\overset{\text{no } y}{h(z)} - \frac{\partial}{\partial z}\overset{\text{no } z}{g(y)}\right)\hat{i} - \left(\frac{\partial}{\partial x}\overset{\text{no } x}{h(z)} - \frac{\partial}{\partial z}\overset{\text{no } z}{f(x)}\right)\hat{j} + \left(\frac{\partial}{\partial x}\overset{\text{no } x}{g(y)} - \frac{\partial}{\partial y}\overset{\text{no } y}{f(x)}\right)\hat{k}$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = 0 \ \checkmark$$

(27) $f, g, h : \mathbb{R}^3 \to \mathbb{R}$

$F(x,y,z) = \left(f(y,z), g(x,z), h(x,y)\right)$ has zero divergence ~ $\nabla \cdot F = 0$

$$\nabla \cdot F = \frac{\partial}{\partial x}\overset{\text{no } x}{f(y,z)} + \frac{\partial}{\partial y}\overset{\text{no } y}{g(x,z)} + \frac{\partial}{\partial z}\overset{\text{no } z}{h(x,y)} = 0 \ \checkmark$$