2/2

MATH 255 — Excellent and Very nice! — WENJIE LIU

Dr. Byungdo Park — 2/24/17

Homework #2

Section 4.4 Exercises 24, 25, 30, 33, 34, 38

Section 7.1 Exercises 9, 10, 12, 16, 18, 19.

Section 4.4

24. Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is a C^2 scalar function. Which of the following expressions are meaningful, and what are nonsense? For those which are meaningful, decide whether the expression defines a scalar function or a vector field.

(a) curl (grad f) → meaningful, the curl of a vector field results in a vector field.

(b) grad (curl f) → nonsense

(c) div (grad f) → meaningful, the divergence of a vector field is a scalar function.

(d) grad (div f) → meaningful, the gradients of a scalar function is a vector field.

(e) curl (div f) → nonsense.

(f) div (curl f) → meaningful, the divergence of a vector field is a scalar function.

25. Suppose $F: \mathbb{R}^3 \to \mathbb{R}^3$ is a C^2 vector field. Which of the following expressions are meaningful, and which are nonsense? For those which are meaningful, decide whether the expression defines a scalar function or a vector field.

(a) curl (grad F) → Nonsense

(b) grad (curl F) → Nonsense

(c) div (grad F) → Nonsense

(d) grad (div F) → meaningful, vector Field.

(e) curl (div F) → Nonsense

(f) div (curl F) → meaningful, scalar function

Verify that $\nabla \times (\nabla f) = 0$ for the functions

30. $f(x,y,z) = xy + yz + xz$

$$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) i - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) j + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) k$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (y+z,\ x+z,\ y+x)$$

$$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x+z & y+x \end{vmatrix} = \left(\frac{\partial}{\partial y}(y+x) - \frac{\partial}{\partial z}(x+z)\right) i - \left(\frac{\partial}{\partial x}(y+x) - \frac{\partial}{\partial z}(y+z)\right) j + \left(\frac{\partial}{\partial x}(x+z) - \frac{\partial}{\partial y}(y+z)\right) k$$

$$= (1-1)\,i - (1-1)\,j + (1-1)\,k = 0$$

Thus, $\nabla \times (\nabla f) = 0$.

33. Show that $F = y(\cos x)\,i + x(\sin y)\,j$ is not a gradient vector field.

If F were a gradient field, then it would satisfy $\text{curl } F = 0$.

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y(\cos x) & x(\sin y) & 0 \end{vmatrix}$$

$$= \left(\cancel{\frac{\partial}{\partial y}(0)} - \frac{\partial}{\partial z}(x(\sin y))\right) i - \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y(\cos x))\right) j + \left(\frac{\partial}{\partial x}(x(\sin y)) - \frac{\partial}{\partial y}(y(\cos x))\right)$$

$$= (0-0)\,i - (0-0)\,j + (\sin y - \cos x)\,k$$

$$= (\sin y - \cos x)\,k$$

$\text{curl } F \neq 0$. so F cannot be a gradient vector field.

34. Show that $F=(x^2+y^2)i - 2xy\, j$ is not a gradient field.

If F were a gradient field, then it would satisfy curl $F=0$.

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$=\left(\frac{\partial}{\partial y}(0)-\frac{\partial}{\partial z}(-2xy)\right)i-\left(\frac{\partial}{\partial x}(0)-\frac{\partial}{\partial z}(x^2+y^2)\right)j+\left(\frac{\partial}{\partial x}(-2xy)-\frac{\partial}{\partial y}(x^2+y^2)\right)k$$

$$=(0-0)i-(0-0)j+((-2y)-(2y))k$$

$$=(-2y-2y)k = -4yk \neq 0.$$

So F cannot be a gradient vector field.

38. Let $\mathbf{r}(x,y,z)=(x,y,z)$ and $r=\sqrt{x^2+y^2+z^2}=\|\mathbf{r}\|$. Prove the following identities.

(a) $\nabla(1/r)=-\mathbf{r}/r^3$; and, in general, $\nabla(r^n)=nr^{n-2}\mathbf{r}$ and $\nabla(\log r)=\mathbf{r}/r^2$? ?

$$\nabla(1/r)=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\left(\frac{1}{r}\right)$$

$$=\left(-\frac{1}{r^2}\frac{dr}{\partial x}\right)i+\left(-\frac{1}{r^2}\frac{dr}{\partial y}\right)j+\left(-\frac{1}{r^2}\frac{dr}{dz}\right)k$$

$$=-\frac{1}{r^2}\left(\frac{\vec{r}}{r}\right)=-\frac{\vec{r}}{r^3}$$

(b) $\nabla^2(1/r)=0$, $r\neq 0$; and, in general, $\nabla^2 r^n = n(n+1)r^{n-2}$?

$$\nabla\left(\frac{1}{r}\right)=-\frac{\mathbf{r}}{r^3}$$

$$\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)=\frac{1}{r^3}\nabla\cdot\mathbf{r}+\mathbf{r}\cdot\nabla\left(\frac{1}{r^3}\right)$$

$$=\frac{3}{r^3}+\mathbf{r}\cdot\left(-\frac{3\mathbf{r}}{r^5}\right)$$

$$=\frac{3}{r^3}-\frac{3}{r^3}=0$$

(c) $\nabla\cdot(\vec{r}/r^3)=0$; and in general, $\nabla\cdot(r^n\vec{r})=(n+3)r^n$?

$\nabla\cdot\left(\frac{r}{r^3}\right)=\frac{1}{r^3}\nabla\cdot r+r\cdot\nabla\left(\frac{1}{r^3}\right)$?

$=\frac{3}{r^3}+r\cdot\left(\frac{-3r}{r^5}\right)$ Please indicate if it is a vector.

$=\frac{3}{r^3}-\frac{3}{r^3}=0$

(d) $\nabla\times r=0$; and in general, $\nabla\times(r^n r)=0$.

$$\nabla\times r=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=\left(\frac{\partial}{\partial y}(z)-\frac{\partial}{\partial z}(y)\right)i-\left(\frac{\partial}{\partial x}(z)-\frac{\partial}{\partial z}(x)\right)j+\left(\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right)k$$

$=(0-0)i-(0-0)j+(0-0)k$

$=0.$

Section 7.1

9. Let $f(x,y,z)=y$ and $c(t)=(0,0,t)$, $0\le t\le 1$. Prove that $\int_c f\,ds=0$.

$\int_c f\,ds=\int_a^b f(x(t),y(t),z(t))\,\|c'(t)\|\,dt$

$c'(t)=(0,0,1)$

$\|c'(t)\|=\sqrt{(0)^2+(0)^2+(1)^2}=\sqrt{1}=1$

$\int_c f\,ds=\int_{t=0}^{t=1}0\,(1)\,dt=0.$

10. Evaluate the following path integrals $\int_c f(x,y,z)\,ds$, where

(a) $f(x,y,z)=x+y+z$ and $c: t\mapsto(\sin t,\cos t,t)$, $t\in[0,2\pi]$

$\int_c f\,ds=\int_a^b f(x(t),y(t),z(t))\,\|c'(t)\|\,dt$

$c'(t)=(\cos t,-\sin t,1)$

$\|c'(t)\|=\sqrt{\cos^2 t+(-\sin^2 t)+1^2}=\sqrt{1+1}=\sqrt{2}$

$\int_c f\,ds=\int_0^{2\pi}(\sin t+\cos t+t)(\sqrt{2})\,dt$

$=\sqrt{2}\int_0^{2\pi}\sin t+\cos t+t\,dt$

$=\sqrt{2}\left[-\cos t+\sin t+\frac{t^2}{2}\right]_0^{2\pi}$

$=\sqrt{2}\left[\left(-\cos(2\pi)+\sin(2\pi)+\frac{(2\pi)^2}{2}\right)-(-\cos(0)+\sin(0)+0)\right]$

$=\sqrt{2}\left[(-1+0+2\pi^2)-(-1+0+0)\right]=\sqrt{2}(-1+2\pi^2+1)=\boxed{2\sqrt{2}\pi^2}$

(b) $f(x,y,z) = \cos z$, c as in part (a)

$$\int_c f\,ds = \int_a^b f(x(t), y(t), z(t))\,\|c'(t)\|\,dt$$

$$c'(t) = (\cos t, -\sin t, 1)$$

$$\|c'(t)\| = \sqrt{\cos^2 t + (-\sin^2 t) + 1^2} = \sqrt{2}$$

$$\int_c f\,ds = \int_0^{2\pi} (\cos t)\sqrt{2}\,dt = \sqrt{2}\int_0^{2\pi} \cos t\,dt$$

$$= \sqrt{2}\left[\sin t\right]_0^{2\pi} = \sqrt{2}\left(\sin(2\pi) - \sin(0)\right) = \boxed{0}$$

12. Evaluate the integral of $f(x,y,z)$ along the path c, where
(a) $f(x,y,z) = x\cos z$, $c: t \mapsto t\mathbf{i} + t^2\mathbf{j}$, $t \in [0,1]$.

$$\int_c f\,ds = \int_a^b f(x(t), y(t), z(t))\,\|c'(t)\|\,dt$$

$$c'(t) = (1, 2t)$$

$$\|c'(t)\| = \sqrt{1^2 + (2t)^2} = \sqrt{1+4t^2}$$

$$\int_c f\,ds = \int_0^1 t\cos(0)\sqrt{1+4t^2}\,dt = \int_0^1 t\sqrt{1+4t^2}\,dt$$

$u = 1+4t^2$ $\quad du = 8t\,dt$ $\quad t=1 \Rightarrow u=5$ $\quad t=0 \Rightarrow u=1$

$$= \frac{1}{8}\int_{u=1}^{u=5} \sqrt{u}\,du = \frac{1}{8}\int_{u=1}^{u=5} u^{\frac{1}{2}}\,du = \frac{1}{8}\left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=5}$$

$$= \frac{1}{12}\left(5^{3/2} - 1^{3/2}\right)$$

They are concerned with the application of the path integral to the problem of defining the average value of a scalar function along a path. Define the number $\frac{\int_c f(x,y,c)\,ds}{l(c)}$ to be the average value of f along c. Here $l(c)$ is the length of the path: $l(c) = \int_c \|c'(t)\|\,dt$.

16. (a) Justify the formula $[\int_c f(x,y,z)\,ds]/l(c)$ for the average value of f along c using Riemann Sums.

The path Integral $\lim_{N\to\infty} S_N = \int_a^b f(x(t), y(t), z(t))\,\|c'(t)\|\,dt = \int_c f(x,y,z)\,ds$

The partial sum has the form $S_N := \sum_{i=1}^{N} f(x(t_i^*), y(t_i^*))(s_i - s_{i-1})$,
where $(s_i - s_{i-1}) = \int_{t_i}^{t_{i+1}} \|c'(t)\|\,dt$.

Then.

$$\sum_{i=1}^{N}(S_i - S_{i-1}) = L(c)$$

and so the quotient

$$\frac{\sum_{i=1}^{N} f(x(t_i^*), y(t_i^*))(S_i - S_{i-1})}{\sum_{i=1}^{N}(S_i - S_{i-1})} = \frac{\sum_{i=1}^{N} f(x(t_i^*), y(t_i^*))(S_i - S_{i-1})}{L(c)}$$

is the approximate average obtained by considering $f(x(t_i^*), y(t_i^*))$ to be constant along each of the arc Δs_i. The limit as $N \to \infty$ of the sequence $\{S_N\}_{N=1}^{\infty}$ is the average value of f along c. So we have the formula $\int_c f(x,y,z)\,ds / L(c)$.

(b) Show that the average value of f along c in Example 1 is $(1 + 4/3\pi^2)$.

$$\frac{\int_c f(x,y,z)\,ds}{L(c)} \quad \text{where } L(c) = \int_c \|c'(t)\|\,dt$$

$$\int_c f(x,y,z)\,ds = \frac{2\sqrt{2}\pi}{3}(3 + 4\pi^2)$$

$$L(c) = \int_0^{2\pi} \|c'(t)\|\,dt = \int_0^{2\pi} \sqrt{2}\,dt$$

$$= \sqrt{2}\,[t]_0^{2\pi} = \sqrt{2}(2\pi - 0) = 2\sqrt{2}\pi.$$

$$\frac{\int_c f(x,y,z)\,ds}{L(c)} = \frac{\frac{2\sqrt{2}\pi}{3}(3+4\pi^2)}{2\sqrt{2}\pi} = \frac{\cancel{2\sqrt{2}\pi}(3+4\pi^2)}{3(\cancel{2\sqrt{2}\pi})}$$

$$= \frac{3}{3} + \frac{4\pi^2}{3} = 1 + 4/3\pi^2$$

(c) In Exercise 10(a) and (b) above, find the average value of f over the given curves.

Ex 10(a) $\dfrac{\int_c f(x,y,z)\,ds}{L(c)} = \dfrac{2\sqrt{2}\pi^2}{\int_0^{2\pi}\sqrt{2}\,dt} = \dfrac{2\sqrt{2}\pi^2}{2\sqrt{2}\pi} = \boxed{\pi}$

Ex 10(b) $\dfrac{\int_c f(x,y,z)\,ds}{L(c)} = \dfrac{0}{\int_0^{2\pi}\sqrt{2}\,dt} = \dfrac{0}{2\sqrt{2}\pi} = \boxed{0}$

18. Suppose the semicircle in Exercise 17 is made of a wire with a uniform density of 2 grams per unit length.

(a) What is the total mass of the wire?

Total mass $= \int_C f \, ds$ uniform density $\Rightarrow f(x,y,z)=1$

$c'(t) = (0, a\cos\theta, -a\sin\theta)$

$\|c'(t)\| = \sqrt{0^2 + (a\cos\theta)^2 + (-a\sin^2\theta)} = \sqrt{0 + a^2\cos^2\theta + a^2\sin^2\theta}$

$= \sqrt{a^2(\cos^2\theta + \sin^2\theta)} = \sqrt{a^2} = a$

Total mass $= \int_0^{\pi} a \, dt$

$= a[t]_0^{\pi} = \boxed{a\pi}$

(b) Where is the center of mass of this configuration of wire?

$\theta \mapsto (0, a\sin\theta, a\cos\theta)$

$x=0 \quad y = a\sin\theta, \quad z = a\cos\theta$

$\bar{x} = \frac{1}{2}\int_0^{\pi} 0(a)\, dt = 0$

$\bar{y} = \frac{1}{2}\int_0^{\pi} (a\sin\theta)\, a\, dt = -\frac{a^2}{2}[\cos\theta]_0^{\pi}$

$= -\frac{a^2}{2}(-1-1) = -\frac{a^2}{2}(-2) = a^2$

$\bar{z} = \frac{1}{2}\int_0^{\pi} (a\cos\theta)\, a\, dt = \frac{a^2}{2}[\sin\theta]_0^{\pi}$

$= \frac{a^2}{2}(0-0) = 0$

The center of mass of this configuration of wire is in $[0, a^2, 0]$

19. Let c be the path given by $c(t) = (t^2, t, 3)$ for $t \in [0,1]$.

(a) Find $L(c)$, the length of the path

$L(c) = \int_C \|c'(t)\| \, dt$

$c'(t) = (2t, 1, 0)$

$\|c'(t)\| = \sqrt{(2t)^2 + (1)^2 + (0)^2} = \sqrt{4t^2+1}$

$L(c) = \int_0^1 \sqrt{4t^2+1}\, dt = \int_0^1 \sqrt{(2t)^2 + 1^2}\, dt$

$= \frac{1}{2}\left[t\sqrt{4t^2+1} + 1\log\left(t + \sqrt{4t^2+1}\right)\right]_0^1$

$= \left(\frac{1}{2}\left[\left(\sqrt{5} + \log(1+\sqrt{5})\right) - (\log 1)\right] = \sqrt{5} \ldots\right.$

(b) Find the average y coordinate along the path c.

$$\frac{\int_0^1 t\sqrt{4t^2+1}\,dt}{L(c)}$$

$L(c) = \frac{\sqrt{5}}{2} + \frac{1}{2}\log(1+\sqrt{5})$

$\int_0^1 t\sqrt{4t^2+1}\,dt$ $\quad u = 4t^2+1 \quad du = 8t\,dt$

$t=1 \Rightarrow u=5 \quad t=0 \Rightarrow u=1$

$= \frac{1}{8}\int_{u=1}^{u=5} \sqrt{u}\,du = \frac{1}{8}\int_{u=1}^{u=5} u^{\frac{1}{2}}\,du$

$= \frac{1}{8}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_{u=1}^{u=5}$

$= \frac{1}{12}\left[5^{3/2}-1\right]$

The average y coordinate $= \dfrac{\frac{1}{12}\left[5^{3/2}-1\right]}{\frac{\sqrt{5}}{2}+\frac{1}{2}\log(1+\sqrt{5})}$