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Math 255
HW#3

2/2 Excellent!

7.2

2) $\vec{F} = y^2\hat{i} + 2xy\hat{j}$ C: $x^2+y^2=1$ Evaluate the line integral $\int_c \vec{F} d\vec{s}$

$$\int_C \vec{F} \cdot ds = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt}\right) dt$$

$\vec{c}(t) = (\cos t, \sin t),\ t \in [0, 2\pi]$

$\vec{c}\,'(t) = (-\sin t, \cos t)$

$\vec{f}(\vec{c}(t)) = (\sin^2 t,\ 2(\cos t \sin t))$

$$\int_0^{2\pi} \left(\sin^2 t(-\sin t) + 2(\cos t \sin t)(\cos t)\right) dt$$

$$= \int_0^{2\pi} \left(-\sin^3 t + 2\cos^2 t \sin t\right) dt = \int_0^{2\pi} \left(\sin t\,(-\sin^2 t + 2\cos^2 t\right) dt$$

$$= -\int_0^{2\pi} \left(\sin t\,(\sin^2 t - 2\cos^2 t)\right) dt = -\int_0^{2\pi} \left(\sin t\,(3\sin^2 t - 2)\right) dt = -\left(3\int_0^{2\pi} \sin^3 t\,dt - 2\int_0^{2\pi} \sin t\,dt\right)$$

Solving for: $3\int \sin^3 t\,dt$: $3\int_0^{2\pi} \sin^2 t \sin t\,dt = 3\int_0^{2\pi} (1-\cos^2 t)\sin t\,dt$

Substitution: $u = \cos t$
$du = -\sin t\,dt$ $\Rightarrow 3\int (1-u^2)\,-du = 3\int (u^2-1)\,du = 3\left[\frac{u^3}{3} - u\right]$

$$3\left[\frac{\cos^3 t}{3} - \cos t\right]_0^{2\pi} = \frac{3\cos^3 t}{3} - 3\cos t = \left[\cos^3 t - 3\cos t\right]_0^{2\pi}$$

Solving for: $2\int \sin t\,dt = -2\left[\cos t\right]_0^{2\pi}$

Then, $-\int_0^{2\pi} \left(\sin t\,(\sin^2 t - 2\cos^2 t)\right) dt = \left[\cos^3 t - 3\cos t - (-\cos t)\right]_0^{2\pi}$

$$= \left[\cos^3 t - \cos t\right]_0^{2\pi} \xrightarrow{\text{plug back}} \int_0^{2\pi} \left(\sin t(-\sin^2 t + \cos^2 t)\right) dt = \left[\cos^3 t - \cos t\right]_0^{2\pi}$$

$$= \left[\cos t - \cos^3 t\right]_0^{2\pi} = \left(\cos(2\pi) - \cos^3(2\pi)\right) - \left(\cos(0) - \cos^3(0)\right)$$
$$(1 - 1) - (1 - 1) = 0$$

$$\boxed{\int_C \vec{F} \cdot ds = 0}$$

4) a) Evaluate $\int_C x\,dy - y\,dx$, $\vec{c}(t) = (\cos t, \sin t)$ $\quad 0 \le t \le 2\pi$

$\vec{c}\,'(t) = (-\sin t, \cos t) \Rightarrow dx = -\sin t$, $dy = \cos t$

$\vec{f}(\vec{c}(t)) = (\cos t - \sin t)$

$$\int_0^{2\pi} \left(\cos t(\cos t) - \sin t(-\sin t)\right) dt = \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = \int_0^{2\pi} dt = t\Big|_0^{2\pi} = 2\pi$$

$$\boxed{\int_C x\,dy - y\,dx = 2\pi}$$

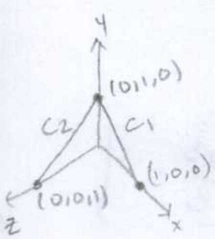
b) Evaluate $\int_C x\,dx + y\,dy$, $\vec{c}(t) = (\cos \pi t, \sin \pi t)$ $\quad 0 \le t \le 2$

$\vec{c}\,'(t) = (-\pi \sin \pi t, \pi \cos \pi t) \Rightarrow dx = -\pi \sin \pi t, \ dy = \pi \cos \pi t$

$\vec{f}(\vec{c}(t)) = (\cos \pi t + \sin \pi t)$

$$\int_0^2 \left(\cos \pi t(-\pi \sin \pi t) + \sin \pi t(\pi \cos \pi t)\right) dt = \int_0^2 -(\pi \cos \pi t \sin \pi t) + (\pi \cos \pi t \sin \pi t)\,dt$$

$$= \int_0^2 0\,dt = 0 \qquad \boxed{\int_C x\,dx + y\,dy = 0}$$



c) Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$, C contains straight line segments joining: $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$

$\vec{c}_1(t) = (1-t, t, 0) \quad 0 \le t \le 1 \qquad c_1'(t) = (-1, 1, 0) \Rightarrow dx = -1, dy = 1, dz = 0$

$\vec{c}_2(t) = (0, 1-t, t) \quad 0 \le t \le 1 \qquad c_2'(t) = (0, -1, 1) \Rightarrow dx = 0, dy = -1, dz = 1$

$\vec{f}(\vec{c}_1(t)) = ((t)(0) + (1-t)(0) + (1-t)(t))$

$\vec{f}(\vec{c}_2(t)) = ((1-t)(t) + (0)(t) + (0)(1-t))$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} \Rightarrow \int_0^1 \left(t(0)(-1) + (1-t)(0)(1) + (1-t)(t)(0)\right) dt + \int_0^1 \left((1-t)(t)(0) + (0)(t)(-1) + (0)(1-t)(1)\right) dt$$

$$= \int_0^1 0\,dt + \int_0^1 0\,dt = 0 + 0 = 0$$

$$\boxed{\int_C yz\,dx + xz\,dy + xy\,dz = 0}$$

d) Evaluate $\int_C x^2dx - xy\,dy + dz$, C = Parabola $z=x^2$, $y=0$ from $(-1,0,1)$ to $(1,0,1)$

$\vec{c}(t) = (t, 0, t^2) \quad -1 \le t \le 1$

$\vec{c}'(t) = (1, 0, 2t) \Rightarrow dx = 1,\ dy = 0,\ dz = 2t$

$\vec{F}(\vec{c}(t)) = (t^2 - t(0) + 1)$

$$\int_{-1}^{1} (t^2(1) - t(0)(0) + 2t)\,dt = \int_0^1 (t^2 + 2t)\,dt = \left[\frac{t^3}{3} + t^2\right]_{-1}^{1} = \underbrace{\left(\frac{1}{3}+1\right)}_{\frac{4}{3}} - \underbrace{\left(\frac{-1}{3}+1\right)}_{\frac{2}{3}} = \frac{2}{3}$$

$$\boxed{\int_C x^2dx - xy\,dy + dz = \frac{2}{3}}$$

6) a) $\vec{F}$ is perpendicular to $\vec{c}'(t)$ at the point $\vec{c}(t)$ Show: $\int_C \vec{F}\cdot d\vec{s} = 0$

Proof: suppose $\vec{c} \in [a,b]$

Since $\vec{c}'(t) \perp \vec{F}(\vec{c}(t))$

then, $\vec{c}'(t) \cdot \vec{F}(\vec{c}(t)) = 0$ (by the lemma)

thus, $\int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t)\,dt = 0$

$$\int_C \vec{F}\cdot dS = 0$$

$\square$

lemma: let $\vec{u}$ and $\vec{v}$ be non-zero vectors, $\vec{u}\cdot\vec{v} = 0 \iff \vec{u} \perp \vec{v}$

($\Leftarrow$) if $\vec{u} \perp \vec{v}$, then $\theta = 90° = \pi/2$

then $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\underbrace{\cos(\pi/2)}_{0}$
$= 0$

Thus, if $\vec{u} \perp \vec{v}$, then $\vec{u}\cdot\vec{v} = 0$

($\Rightarrow$) suppose $\vec{u}\cdot\vec{v} = 0$

then, $\|\vec{u}\|\|\vec{v}\|\cos\theta = \frac{0}{\|\vec{u}\|\|\vec{v}\|}$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pi/2$

Thus, if $\vec{u}\cdot\vec{v} = 0$, then $\vec{u} \perp \vec{v}$.

$\square$

b) $\vec{F}$ is parallel to $\vec{c}'(t)$ at $\vec{c}(t)$ show: $\int_C \vec{F}\cdot ds = \int_C \|\vec{F}\|\,ds$

Proof: Suppose $\vec{c} \in [a,b]$

Since $\vec{c}'(t) \parallel \vec{F}(\vec{c}(t))$

Then, $\vec{c}'(t) \cdot \vec{F}(\vec{c}(t)) = \|\vec{c}'(t)\|\,\|\vec{F}(\vec{c}(t))\|$
$= \|\vec{F}\|$ (by the lemma)

thus, $\int_a^b \|\vec{F}(\vec{c}'(t))\|\|\vec{c}'(t)\|\,dt = \int_a^b \|\vec{F}\|\,dt$

$$\int_C \vec{F}\cdot ds = \int_C \|\vec{F}\|\,ds$$

$\square$

Lemma: let $\vec{u}$ and $\vec{v}$ be non-zero vectors, $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\| \iff \vec{u} \parallel \vec{v}$

($\Leftarrow$) if $\vec{u} \parallel \vec{v}$, then $\theta = 0° = 0$

then $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\underbrace{\cos(0)}_{1} = \|\vec{u}\|\|\vec{v}\|$

Thus, if $\vec{u} \parallel \vec{v}$, then $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|$

($\Rightarrow$) suppose $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 0$

Thus, if $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|$, then $\vec{u} \parallel \vec{v}$.

$\square$

7) Suppose path $\vec{c}$ has length ℓ, $\|\vec{F}\| \leq M$, Prove: $\left|\int_c \vec{F}\cdot ds\right| \leq M\ell$

Proof: let $\vec{c} \in [a,b]$

Then, $\left|\int_c \vec{F}\cdot ds\right| = \left|\int_a^b (\vec{F}(\vec{c}(t))\cdot \vec{c}'(t))\,dt\right|$

$\Rightarrow \left|\int_a^b (\vec{F}(\vec{c}(t))\cdot \vec{c}'(t))\,dt\right| \leq \int_a^b \|\vec{F}(\vec{c}(t))\cdot \vec{c}'(t)\|\,dt$

$\Rightarrow \left|\int_a^b (\vec{F}(\vec{c}(t))\cdot \vec{c}(t))\,dt\right| \leq \int_a^b \|\vec{F}(\vec{c}(t))\|\|\vec{c}'(t)\|\,dt$

$\leq M\int_a^b \|c'(t)\|\,dt$

$\Rightarrow \left|\int_a^b (\vec{F}(\vec{c}(t))\cdot \vec{c}(t))\,dt\right| \leq M\ell$

Thus, $\left|\int_c \vec{F}\cdot ds\right| \leq M\ell$

$\square$

8) Evaluate $\int_c \vec{F}\cdot ds$, $\vec{F}(x,y,z) = y\hat{i} + 2x\hat{j} + y\hat{k}$, $\vec{c}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ $\quad 0 \leq t \leq 1$

$\vec{c}'(t) = (1, 2t, 3t^2)$

$\vec{F}(\vec{c}(t)) = (t^2 + 2t + t^2)$

$$\int_c \vec{F}\cdot ds = \int_0^1 (t^2(1) + 2t(2t) + t^2(3t^2))\,dt = \int_0^1 (t^2 + 4t^2 + 3t^4)\,dt = \int_0^1 (5t^2 + 3t^4)\,dt$$

$$= \left[\frac{5t^3}{3} + \frac{3t^5}{5}\right]_0^1 = \frac{5}{3} + \frac{3}{5} = \frac{25}{15} + \frac{9}{15} = \frac{34}{15}$$

$$\boxed{\int_c \vec{F}\cdot ds = \frac{34}{15}}$$

11) Evaluate $\int_C \vec{F}\cdot ds$, $\vec{F}(x,y,z)=x\hat{i}+y\hat{j}$, $t \mapsto (\cos^3 t, \sin^3 t)$ $0 \le t \le 2\pi$

$\vec{c}(t)=(\cos^3 t, \sin^3 t)$

$\vec{c}'(t)=(-3\cos^2 t\sin t, 3\sin^2 t\cos t)$

$\vec{F}(\vec{c}(t))=(\cos^3 t + \sin^3 t)$

$$\int_C \vec{F}\cdot ds = \int_0^{2\pi} (\cos^3 t(-3\cos^2 t\sin t) + \sin^3 t(3\sin^2 t\cos t))dt$$

$$= \int_0^{2\pi} (-3\cos^5 t\sin t + 3\sin^5 t\cos t)dt$$

$$= \int_0^{2\pi} (3\cos t\sin t(\sin^4 t - \cos^4 t))dt = \frac{3}{2}\int_0^{2\pi} (\sin 2t(\sin^2 t + \cos^2 t)(\sin^2 t - \cos^2 t))dt$$

$$= \frac{3}{2}\int_0^{2\pi} (\sin 2t(-\cos 2t))dt = \frac{3}{4}\int_0^{2\pi} \sin 4t\, dt$$

Substitution:
$u = 4t$
$dx = \frac{1}{4}du$

$$= \frac{3}{16}\int \sin u\, du = \frac{3}{16}\left[-\cos u\right] = \frac{3}{16}\left[-\cos 4x\right]_0^{2\pi}$$

$$= \frac{3}{16}(-\cos(8\pi)+\cos(0)) = \frac{3}{16}(-1+1) = \frac{3}{16}(0) = 0$$

$$\boxed{\int_C \vec{F}\cdot ds = 0}$$

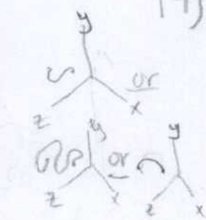
13) $\vec{c}(t)$ is a path, $\vec{T}$ is the unit tangent vector, evaluate $\int_C \vec{T}\cdot ds$

by the definition of a unit vector, $\vec{T} = \dfrac{c'(t)}{\|c'(t)\|}$

Then, $\int_C \vec{T}\cdot ds = \int \left(\dfrac{c'(t)}{\|c'(t)\|}\cdot c'(t)\right)dt$

$$= \int \frac{(c'(t))^2}{\|c'(t)\|}dt = \int \frac{\|c'(t)\|^{\cancel{2}}}{\cancel{\|c'(t)\|}} = \int \|c'(t)\|dt$$

$$\boxed{\int_C \vec{T}\cdot ds = \int \|c'(t)\|dt}$$



17) Evaluate $\int_C 2xyz\,dx + x^2 z\,dy + x^2 y\,dz$ C: Simple curve connecting $(1,1,1)$ to $(1,2,4)$

Several paths can be taken to obtain a simple curve from $(1,1,1) \to (1,2,4)$

To compute a result independent of path, we must assume $\vec{F} = \nabla f$ and integrate over the gradient field.

$\vec{F}(x,y,z) = (2xyz, x^2 z, x^2 y)$

if $\vec{F} = \nabla f$, $\nabla f = \frac{Fx}{\partial x} + \frac{Fy}{\partial y} + \frac{Fz}{\partial z} = 2xyz + x^2 z + x^2 y$

then, $f = x^2 y z \Rightarrow \left(\frac{x^2yz}{\partial x} = 2xyz\right) + \left(\frac{x^2yz}{\partial y} = x^2 z\right) + \left(\frac{x^2yz}{\partial z} = x^2 y\right)$

$$\int_C \vec{F} \cdot ds = \int_C \nabla f \cdot ds$$

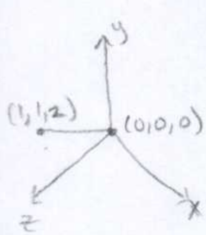
$$= \int_a^b \left(\frac{Fx}{\partial x} + \frac{Fy}{\partial y} + \frac{Fz}{\partial z}\right) dt \Rightarrow f(b) - f(a) \Rightarrow f(1,2,4) - f(1,1,1)$$

$$f(1,2,4) - f(1,1,1) = \left((1^2)(2)(4)\right) - \left((1^2)(1)(1)\right) = 8 - 1 = 7$$

$$\boxed{\int_C 2xyz\,dx + x^2 z\,dy + x^2 y\,dz = 7}$$

18) Suppose $\nabla f(x,y,z) = 2xyze^{x^2}\,\hat{i} + ze^{x^2}\,\hat{j} + ye^{x^2}\,\hat{k}$ if $f(0,0,0) = 5$

find $f(1,1,2) = ?$



$\nabla f = \vec{F}$ $\nabla f = 2xyze^{x^2} + ze^{x^2} + ye^{x^2} = \vec{F}$

$\vec{c}(t) = (t, t, 2t)$ $0 \le t \le 1$

$\vec{c}\,'(t) = (1,1,2)$ $0 \le t \le 1$

$\vec{F}(\vec{c}(t)) = \left(2(t)(t)(2t)e^{t^2} + 2te^{t^2} + te^{t^2}\right) = \left(4t^3 e^{t^2} + 2te^{t^2} + te^{t^2}\right)$

$$\int_C \vec{F} \cdot ds = \int_0^1 \left(4t^3 e^{t^2}(1) + 2te^{t^2}(1) + te^{t^2}(2)\right) dt = \int_0^1 4t^3 e^{t^2} + 2te^{t^2} + 2te^{t^2}$$

$$= \int_0^1 (4t^3 e^{t^2} + 4te^{t^2})\,dt = 4\int_0^1 (t^3 e^{t^2} + te^{t^2})\,dt = 4\int_0^1 t^3 e^{t^2}\,dt + 4\int_0^1 te^{t^2}\,dt$$

$4\int te^{t^2}dt \Rightarrow$ integration by parts: $\int fg' = fg - \int f'g$ $\quad f = e^{t^2} \quad g' = t$

$f' = 2te^{t^2} \quad g = \frac{t^2}{2}$

$\Rightarrow 4\int te^{t^2}dt = 4(e^{t^2})\left(\frac{t^2}{2}\right) - 4\int (2te^{t^2})\frac{t^2}{2}dt = \frac{4(e^{t^2}t^2)}{2} - 4\int (\cancel{2}te^{t^2})\left(\frac{t^2}{\cancel{2}}\right)dt$

$\Rightarrow 4\int te^{t^2}dt = 2t^2e^{t^2} - 4\int t^3e^{t^2}dt$

returning to the original problem, $\int_C \vec{F}\cdot ds = 4\int_0^1 (t^3e^{t^2})dt + 4\int_0^1 te^{t^2}dt$

$= 4\cancel{\int_0^1 t^3e^{t^2}dt} - 4\cancel{\int_0^1 t^3e^{t^2}dt} + 2t^2e^{t^2} = 2t^2e^{t^2}$

$\left[2t^2e^{t^2}\right]_0^1 = 2(1)^2e^{(1)^2} = 2e$

$\int_C \vec{F}\cdot ds = \int_C \nabla f\, ds = f(b) - f(a) \Rightarrow f(1,1,2) - f(0,0,0) = 2e$

$\Rightarrow f(1,1,2) - 5 = 2e$

$\Rightarrow f(1,1,2) = 2e + 5$

$\boxed{f(1,1,2) = 2e+5}$