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MATH-255

Excellent! 2/2

$\boxed{7.3}$ - # 1, 2, 5, 6, 9, 11, 12, 15, 16, 19, 20

1) $\vec{T}_u = \left(\frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0) \right)$

$$\left.\begin{array}{l} 0 = 2u \\ 1 = u^2 + v \\ 1 = v^2 \end{array}\right\} \Rightarrow \begin{array}{l} u = 0 \\ v = 1 \end{array}$$

$\vec{T}_u = (2, 2u, 0)$

$\vec{T}_v = \left(\frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0) \right)$

$\vec{T}_v = (0, 1, 2v)$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & 2u & 0 \\ 0 & 1 & 2v \end{vmatrix} = (4uv, -4v, 2)$$

$\Downarrow$ at $(0,1)$

$(0, -4, 2)$

Tangent plane $= -4(y-1) + 2(z-1) = 0$

2) $-\frac{1}{4} = u^2 - v^2 = (u+v)(u-v) = \frac{1}{2}u - \frac{1}{2}v = -\frac{1}{4} = 4u - 4v = -2$

$\frac{1}{2} = u + v$

$2 = u^2 + 4v$

$$\begin{array}{r} 4u - \cancel{4v} = -2 \\ (+)\ u^2 + \cancel{4v} = 2 \\ \hline u(u+4) = 0 \\ u = 0 \text{ or } u = -4 \end{array}$$

If $u = 0$, then $v = \frac{1}{2}$, and our system of eqns make sense, so we have $(u, v) = (0, \frac{1}{2})$

$\vec{T}_u = (2u, 1, 2u) \xrightarrow{(0,\frac{1}{2})} (0, 1, 0)$

$\vec{T}_v = (-2v, 1, 4) \xrightarrow{(0,\frac{1}{2})} (-1, 1, 4)$

$\vec{n} = (4, -0, 1) = (4, 0, 1)$

So, $\vec{n} \cdot \left(x+\frac{1}{4},\ y-\frac{1}{2},\ z-2\right) = 0$

$\Rightarrow 4\left(x+\frac{1}{4}\right) + (z-2) = 0$

$= 4x + 1 + z - 2 = 0$

$= 4x + z - 1 = 0$

5) $\Phi(u_0, v_0)$ is not regular if $\vec{T}_u \times \vec{T}_v = \vec{0}$.

$\vec{T}_u = (2u,\ 2u,\ 0)$, $\vec{T}_v = (-2v,\ 2v,\ 1)$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 2u & 0 \\ -2v & 2v & 1 \end{vmatrix} = (2u,\ -2u,\ 4uv + 4uv) = (2u, -2u, 8uv)$$

So, $\vec{T}_u \times \vec{T}_v = \vec{0}$ where $u = 0$, and at any v.

So, it is not regular when $u = 0$

6) $\vec{T}_u = (1,\ 1,\ 2v)$, $\vec{T}_v = (-1,\ 1,\ 2u)$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2v \\ -1 & 1 & 2u \end{vmatrix} = \left(2u - 2v,\ -(2u+2v),\ 1 + 1\right)$$

$$= \left(2(u-v),\ -2(u+v),\ 2\right)$$

~~If $u=0$: $(-2v, -2v, 0)$.~~

~~So, $v=0$ if $\vec{T}_u \times \vec{T}_v = (0,0,0)$ and $u=0$.~~

~~It is not regular at $(0,0)$.~~

$\vec{T}_u \times \vec{T}_v$ is never $(0,0,0)$ because it will always have 2 at the $\hat{k}$ vector. So, it is always regular.

9) $\vec{T}_u = (\cos v \cos u, \sin v \cos u, -\sin u)$

$\vec{T}_v = (-\sin v \sin u, \cos v \sin u, 0)$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v \cos u & \sin v \cos u & -\sin u \\ -\sin v \sin u & \cos v \sin u & 0 \end{vmatrix}$$

$$= (0 + \cos v \sin^2 u, \sin v \sin^2 u, \cos^2 v \sin u \cos u + \sin^2 v \sin u \cos u)$$

$$= (\cos v \sin^2 u, \sin v \sin^2 u, \sin u \cos u)$$

$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{\cos^2 v \sin^4 u + \sin^2 v \sin^4 u + \sin^2 u \cos^2 u}$$

$$= \sqrt{\sin^4 u + \sin^2 u \cos^2 u}$$

$$= \sqrt{\sin^2 u (\cancelto{1}{\sin^2 u + \cos^2 u})}$$

$$= \sqrt{\sin^2 u} = \sin u$$

$$\text{unit } \vec{n} = \frac{1}{\sin u}(\cos v \sin^2 u, \sin v \sin^2 u, \sin u \cos u)$$

$$= (\cos v \sin u, \sin v \sin u, \cos u)$$

This surface is a unit sphere at the origin

11) $\vec{T}_u = (0, 1, 0)$

$\vec{T}_v = (\cos v, 0, -\sin v)$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ \cos v & 0 & -\sin v \end{vmatrix} = (-\sin v, 0, -\cos v)$$

$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{\sin^2 v + \cos^2 v} = 1$$

$$\vec{n} = (-\sin v, 0, -\cos v)$$

Our unit normal vector is analogous to cylindrical coordinates $r^2 = x^2 + z^2$, a cylinder about the y-axis. So our surface is a cylinder. (treat v like θ, and u like y).

12) $\vec{T}_u = (-(2-\cos v)\sin u,\ (2-\cos v)\cos u,\ 0)$

$= (-2\sin u + \sin u\cos v,\ 2\cos u - \cos u\cos v,\ 0)$

$\vec{T}_v = (\sin v\cos u,\ \sin v\sin u,\ \cos v)$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u + \sin u\cos v & 2\cos u - \cos u\cos v & 0 \\ \sin v\cos u & \sin v\sin u & \cos v \end{vmatrix}$$

$$= (2\cos u\cos v - \cos u\cos^2 v,\ -(-2\sin u\cos v + \sin u\cos^2 v),\ -2\sin v + \cos v\sin v)$$

$$= (\cos u\cos v(2-\cos v),\ \sin u\cos v(2-\cos v),\ -\sin v(2-\cos v))$$

$$= (2-\cos v)(\cos u\cos v,\ \sin u\cos v,\ -\sin v)$$

$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{\cos^2 u\cos^2 v(2-\cos v)^2 + \sin^2 u\cos^2 v(2-\cos v)^2 + \sin^2 v(2-\cos v)^2}$$

$$= \sqrt{\cos^2 v(2-\cos v)^2 + \sin^2 v(2-\cos v)^2}$$

$$= \sqrt{(2-\cos v)^2} = 2-\cos v$$

$$\vec{n} = \frac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|} = (\cos u\cos v,\ \sin u\cos v,\ -\sin v)$$

It is ~~[illegible]~~ regular everywhere, b/c if $\cos u = 0$,

then $\sin u = 1$ and vice versa. So it is never $(0,0,0)$.

15) Since z is of the form $z = g(x,y)$, we can let
$x = u$, $y = v$, $z = g(u,v) = 3u^2 + 8uv$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial g}{\partial u} \\ 0 & 1 & \frac{\partial g}{\partial v} \end{vmatrix} = \left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right)$$

$$-\frac{\partial g}{\partial u} = -(6u + 8v) = -6u + -8v = -6u - 8v$$

$$-\frac{\partial g}{\partial v} = -(8u) = -8u$$

So, $\vec{T}_u \times \vec{T}_v = (-6u - 8v, -8u, 1) \xrightarrow{\text{at } (1,0,3)} (-6, -8, 1)$

Tan. plane at $(1, 0, 3)$:

$$-6(x-1) - 8(y) + (z-3) = 0$$
$$-6x + 6 - 8y + z - 3 = 0$$
$$z - 6x - 8y + 3 = 0$$

16) $z = \sqrt{2 - x^3 - 3xy}$,
Let $x = u$, $y = v$, $z = g(u,v) = \sqrt{2 - u^3 - 3uv}$

$$-\frac{\partial g}{\partial u} = -\left(\frac{1}{2}(2 - u^3 - 3uv)^{-1/2}\left(\overset{(-3u^2-3v)}{\cancel{\quad}}\right)\right) = -\frac{\overset{(-3u^2-3v)}{\cancel{\quad}}}{2}(2 - u^3 - 3uv)^{-1/2}$$

$$-\frac{\partial g}{\partial v} = -\left(\frac{1}{2}(\overset{2}{\cancel{\quad}} - u^3 - 3uv)^{-1/2}(-3u)\right) = \frac{3u}{2}(2 - u^3 - 3uv)^{-1/2}$$

At $(1, \frac{1}{3}, 0)$,

$\vec{T}_u \times \vec{T}_v = \left(\frac{3u^2 + 3v}{2\sqrt{2 - u^3 - 3uv}}, \frac{3u}{2\sqrt{2 - u^3 - 3uv}}, 1\right)$ results in a zero denominator, so we can multiply by $2\sqrt{2 - u^3 - 3uv}$. Also notice $\vec{n}$ need not be unit vector in this case of finding tangent planes, so our multiplication will still work.

$\underbrace{\left(2\sqrt{2-u^2-3uv}\right)\left(\vec{T}_u \times \vec{T}_v\right)}_{\text{call this } \vec{n}} = \left(3u^2+3v,\ 3u,\ 2\sqrt{2-u^2-3uv}\right)$

$$\vec{n} \text{ at } \left(1, \tfrac{1}{3}, 0\right) = (4, 3, 0)$$

So our tan plane is

$$4(x-1) + 3\left(y-\tfrac{1}{3}\right) = 0$$

$$4x - 4 + 3y - 1 = 0$$

$$4x + 3y - 5 = 0$$

If we have level set $f(x,y,z) = x^3 + 3xy + z^2 - 2 = 0$

We know that $\nabla f(x,y,z)$ is the normal vector orthogonal to the surface (p. 138). So,

$$\nabla f = \left(3x^2+3y,\ 3x,\ 2z\right),$$

and at $\left(1, \tfrac{1}{3}, 0\right)$, we get $(4, 3, 0)$, which is the same normal vector that we obtained previously.

19a) $z^2 = x^2 + y^2 - 5^2$

$\to x^2 + y^2 = z^2 + 25$. Notice, this is of the form $\sqrt{x^2+y^2} = \sqrt{z^2+25}$, where we can treat $\sqrt{z^2+25}$ as r

So, we can have cylindrical coordinate parametrization,

$$\Phi(z, \theta) = \left(\cos\theta\sqrt{z^2+25},\ \sin\theta\sqrt{z^2+25},\ z\right),$$

where $0 \le \theta \le 2\pi$

19b) $\vec{T}_z = \left(z(z^2+25)^{-1/2}\cos\theta,\ z(z^2+25)^{-1/2}\sin\theta,\ 1 \right)$

$\vec{T}_\theta = \left(-\sin\theta(z^2+25)^{1/2},\ \cos\theta(z^2+25)^{1/2},\ 0 \right)$

$$\vec{T}_z \times \vec{T}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ z\cos\theta(z^2+25)^{-1/2} & z\sin\theta(z^2+25)^{-1/2} & 1 \\ -\sin\theta(z^2+25)^{1/2} & \cos\theta(z^2+25)^{1/2} & 0 \end{vmatrix}$$

$$= -\cos\theta(z^2+25)^{1/2}\hat{i} - \left(\sin\theta(z^2+25)^{1/2}\right)\hat{j} + z\cancel{(\cos^2\theta+\sin^2\theta)}\hat{k}$$

$$= \left(-\cos\theta\sqrt{z^2+25},\ -\sin\theta\sqrt{z^2+25},\ z \right)$$

$$\|\vec{T}_z \times \vec{T}_\theta\| = \sqrt{\cos^2\theta(z^2+25) + \sin^2\theta(z^2+25) + z^2}$$

$$= \sqrt{2z^2+25}$$

$$\vec{n} = \frac{1}{\sqrt{2z^2+25}}\left(-\cos\theta\sqrt{z^2+25},\ -\sin\theta\sqrt{z^2+25},\ z \right)$$

19c) At $\Phi(0,\theta)$, $\vec{n} = \frac{1}{5}(-5\cos\theta,\ -5\sin\theta,\ 0) = (-\cos\theta, -\sin\theta, 0)$

$$-\cos\theta(x-x_0) - \sin\theta(y-y_0) = 0$$

Since at $\Phi(0,\theta) = (\underset{x_0}{\underset{\shortparallel}{5\cos\theta}},\ \underset{y_0}{\underset{\shortparallel}{5\sin\theta}},\ \underset{z_0}{\underset{\shortparallel}{0}})$,

$$-\frac{x_0}{5}(x-x_0) - \frac{y_0}{5}(y-y_0) = 0$$

$$-x_0(x-x_0) - y_0(y-y_0) = 0$$

$$x_0(x-x_0) + y_0(y-y_0) = 0$$

$$x_0x - x_0^2 + y_0y - y_0^2 = 0$$

$$x_0x + y_0y = x_0^2 + y_0^2$$

$$x_0x + y_0y = 25$$

19d) ~~At $(-y_0, x_0, 5)$, ... $(-y_0)^2 + (x_0^2) - 5^2$, which equals $x_0^2 + y_0^2$~~

At point $(x_0, y_0, 0)$,

$(x_0 + y_0,\ y_0 - x_0,\ 0 - 5)$

↳ plug in to $x^2 + y^2 - z^2 = 25$ to get

$$x_0^2 + \cancel{2x_0y_0} + y_0^2 + y_0^2 - \cancel{2x_0y_0} + x_0^2 - 25 = 25$$

$$2x_0^2 + 2y_0^2 = 50$$

$$x_0^2 + y_0^2 = 25$$

$\vec{n} = \left(-\frac{x_0}{5}, \frac{-y_0}{5}, 0\right)$, as we saw in part (c).

So, $\left(-\frac{x_0}{5}, \frac{-y_0}{5}, 0\right) \cdot (x + y_0, y - x_0, z - 5) = \frac{-x_0(x+y_0)}{5} + \frac{-y_0(x_0+y)}{5} = 25$

↓

$$-x_0x + \cancel{x_0y_0} - \cancel{x_0y_0} - y_0y = 25$$

For line $(x_0, y_0, 0) + t(y_0, -x_0, 5)$ we do the same.

$(x_0 - y_0,\ y_0 + x_0,\ 5)$

↳ $x_0^2 - \cancel{2x_0y_0} + y_0^2 + y_0^2 + \cancel{2x_0y_0} + x_0^2 - 25 = 25$

$$2x_0^2 + 2y_0^2 = 50$$

$$x_0^2 + y_0^2 = 25$$

$$\left(\frac{-x_0}{5}, \frac{-y_0}{5}, 0\right) \cdot (x - y_0, y + x_0, z - 5) = 25$$

$$\frac{-x_0(x - y_0)}{5} - \frac{y_0(y + x_0)}{5} = 25$$

$$-x_0x + \cancel{x_0y_0} - y_0y - \cancel{x_0y_0} = 25$$

$$x_0x + y_0y = 25$$

20a) $\vec{D}\vec{\Phi}(u_0, v_0) = \begin{pmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \\ \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{pmatrix}$

$\text{Span}(\vec{T}_u, \vec{T}_v) = c_1\vec{T}_u + c_2\vec{T}_v$ for any c_1 + c_2

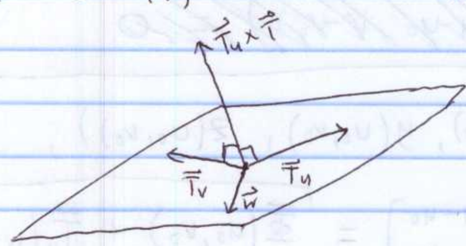
$= c_1 \frac{\partial \Phi}{\partial u} + c_2 \frac{\partial \Phi}{\partial v}$

$\text{range}(\vec{D}) = \text{Span}\Big(\underbrace{\frac{\partial x}{\partial u}\vec{v}_1 + \frac{\partial y}{\partial u}\vec{v}_2 + \frac{\partial z}{\partial u}\vec{v}_3}_{T_u}, \underbrace{\frac{\partial x}{\partial v}\vec{v}_1 + \frac{\partial y}{\partial v}\vec{v}_2 + \frac{\partial z}{\partial v}\vec{v}_3}_{\vec{T}_v}\Big)$

OK, You meant $\vec{v}_1 = \hat{i}$, $\vec{v}_2 = \hat{j}$, $\vec{v}_3 = \hat{k}$.

$= \text{Span}(\vec{T}_u, \vec{T}_v)$

20b) If $\vec{w} \in \text{range}(\vec{D}) = \text{Span}(\vec{T}_u, \vec{T}_v)$, then $\vec{w}$ is also in span $(\vec{T}_u, \vec{T}_v)$. Any vector in the span $(\vec{T}_u, \vec{T}_v)$ is in the same plane as $\vec{T}_u$ + $\vec{T}_v$. Since $(\vec{T}_u \times \vec{T}_v)$ by definition is orthogonal to $\vec{T}_u$ and $\vec{T}_v$, $\vec{w}$ is then $\perp$ to $\vec{T}_u \times \vec{T}_v$



the other direction (converse?)

(over) →

$$\begin{array}{cc} & 3 \times 2 \qquad\qquad\qquad 2 \times 1 \end{array}$$

2oc) $$\vec{D}\vec{\Phi}\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \\ \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{bmatrix}\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x}{\partial u}(u_0,v_0)(u-u_0) + \frac{\partial x}{\partial v}(u_0,v_0)(v-v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0)(u-u_0) + \frac{\partial y}{\partial v}(u_0,v_0)(v-v_0) \\ \frac{\partial z}{\partial u}(u_0,v_0)(u-u_0) + \frac{\partial z}{\partial v}(u_0,v_0)(v-v_0) \end{bmatrix}$$

$$= \vec{T}_u(u-u_0) + \vec{T}_v(v-v_0) = \cancel{(\vec{T}_u, \vec{T}_v)\cdot(u-u_0, v-v_0)}$$

~~Since $\vec{n} = \vec{T}_u \times \vec{T}_v$ and we know that for any vectors $(a \times b)\cdot b = \vec{0}$,~~

~~$\vec{n}\cdot\vec{T}_v = 0, \quad \vec{n}\cdot\vec{T}_u = 0$~~

~~So, $\vec{n}\cdot(\vec{T}_u, \vec{T}_v)\cdot(u-u_0, v-v_0) = 0$~~

Since $\Phi(u_0,v_0) = (x(u_0,v_0),\ y(u_0,v_0),\ z(u_0,v_0))$,

linear approximation as in Ch. 2

$$\vec{\Phi}(u_0,v_0) + \vec{D}\vec{\Phi}(u_0,v_0)\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix} = \vec{\Phi}(u_0,v_0) + \vec{T}_u(u-u_0) + \vec{T}_v(v-v_0)$$

$$\cancel{= (\vec{T}_u, \vec{T}_v)\cdot(u-u_0, v-v_0) + (x(u_0,y_0), y(u_0,y_0), z(u_0,v_0))}$$

$$\cancel{\vec{n}\cdot\left[(\vec{T}_u, \vec{T}_v)\cdot(u-u_0, v-v_0) + (x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))\right]}$$

$$\cancel{= 0 + n_1 x(u_0,v_0) + n_2 y(u_0,v_0) + n_3 z(u_0,v_0)}$$