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HW4: 7.3
MATH 25500
Professor Park

Find an equation for the plane tangent to the given surface at the specified point:

1. $x=2u,\ y=u^2+v,\ z=v^2$, at $(0,1,1)$

$$\vec{R}(x,y,z)=(2u,\ u^2+v,\ v^2)$$
$$\vec{T}_u=\left(\frac{dx}{du},\frac{dy}{du},\frac{dz}{du}\right)=(2,2u,0)$$
$$\vec{T}_v=\left(\frac{dx}{dv},\frac{dy}{dv},\frac{dz}{dv}\right)=(0,1,2v)$$

Normal vector:

$$\vec{T}_u\times\vec{T}_v=\begin{vmatrix} i & j & k\\ 2 & 2u & 0\\ 0 & 1 & 2v\end{vmatrix}$$

$$=(2v2u,\ -4v,\ 2)$$

$2u=0 \quad u^2+v=1 \quad v^2=1$
$\hookrightarrow u=0 \quad u=0 \quad \hookrightarrow v=1$
$\hookrightarrow v=1$

At $(0,1,1)$, $\vec{T}_u\times\vec{T}_v=(0,-4,2)$

Tangent Plane:

$$0(x-0)-4(y-1)+2(z-1)=0$$
$$-4y+4+2z-2=0$$
$$2y-z=1$$

2. $x=u^2-v^2,\ y=u+v,\ z=u^2+4v$, at $\left(-\frac{1}{4},\frac{1}{2},2\right)$

$$\vec{R}(x,y,z)=(u^2-v^2,\ u+v,\ u^2+4v)$$
$$\vec{T}_u=(2u,1,2u)$$
$$\vec{T}_v=(-2v,1,4)$$

Normal vector:

$$\vec{T}_u\times\vec{T}_v=\begin{vmatrix} i & j & k\\ 2u & 1 & 2u\\ 2v & 1 & 4\end{vmatrix}$$

$$=(4-2u,\ 2v2u-8u,\ 2u-2v)$$

$u^2-v^2=-\frac{1}{4} \rightarrow u^2-v^2+\frac{1}{4}=0$ — ①
$u+v=\frac{1}{2} \rightarrow u=\frac{1}{2}-v$ — ②
$u^2+4v=2 \rightarrow u^2+4v-2=0$ — ③

① − ③

$$-v^2+\frac{1}{4}-4v+2=0$$
$$v^2+4v-\frac{9}{4}=0$$
$$4v^2+16v-9=0$$
$$(2v+9)(2v-1)=0$$
$$v=-\frac{9}{2} \text{ or } v=\frac{1}{2}$$

Sub $v=-\frac{9}{2}$ into ②
$$u=\frac{1}{2}-\left(-\frac{9}{2}\right)=\frac{10}{2}=5$$

Sub $v=\frac{1}{2}$ into ②
$$u=\frac{1}{2}-\frac{1}{2}=0$$

$v=\frac{1}{2}$, $u=0$ is the only solution that satisfies equations ①, ②, and ③

At $\left(-\frac{1}{4},\frac{1}{2},2\right)$, $\vec{T}_u\times\vec{T}_v=(4,0,-1)$

Tangent Plane:

$$4\left(x+\frac{1}{4}\right)+0\left(y-\frac{1}{2}\right)-1(z-2)=0$$
$$4x+1-z+2=0$$
$$z-4x=3$$

2/2 Excellent!

Find all points (u_0,v_0), where $S=\Phi(u_0,v_0)$ is not smooth (regular)

5. $\Phi(u,v)=(u^2-v^2,\ u^2+v^2,\ v)$

$$\vec{T}_u=(2u,2u,0)$$
$$\vec{T}_v=(-2v,2v,1)$$

Normal vector:

$$\vec{T}_u\times\vec{T}_v=\begin{vmatrix} i & j & k\\ 2u & 2u & 0\\ -2v & 2v & 1\end{vmatrix}$$

$$=(2u,\ -2u,\ 2v2u+2v2u)$$
$$=2ui-2uj+(2v2u+2v2u)k$$

A surface is smooth (regular) if $\vec{T}_u\times\vec{T}_v\neq 0$

If $u=0$, $\vec{T}_u\times\vec{T}_v=0$

$\hookrightarrow$ At $\Phi(0,v)$ the surface is not regular

6. $\Phi(u,v)=(u-v,\ u+v,\ 2uv)$

$$\vec{T}_u=(1,1,2v)$$
$$\vec{T}_v=(-1,1,2u)$$

Normal vector:

$$\vec{T}_u\times\vec{T}_v=\begin{vmatrix} i & j & k\\ 1 & 1 & 2v\\ -1 & 1 & 2u\end{vmatrix} \ -1$$

$$=(2u-2v,\ -2v-2u,\ 2)$$
$$=(2u-2v)i-(2v+2u)j+2k$$

Since $\vec{T}_u\times\vec{T}_v$ is never 0, the surface is smooth (regular)

Find an expression for a unit vector normal to the surface

9. $x=\cos v\sin u,\ y=\sin v\sin u,\ z=\cos u$ at the image of a point (u,v)* for u in $[0,\pi]$ and v in $[0,2\pi]$. Identify this surface.

$$\vec{T}_u=(\cos v\cos u,\ \sin v\cos u,\ -\sin u)$$
$$\vec{T}_v=(-\sin v\sin u,\ \cos v\sin u,\ 0)$$

Normal vector:

$$\vec{T}_u\times\vec{T}_v=\begin{vmatrix} i & j & k\\ \cos v\cos u & \sin v\cos u & -\sin u\\ -\sin v\sin u & \cos v\sin u & 0\end{vmatrix}$$

$$=(\cos v\sin^2 u,\ \sin v\sin^2 u,\ \cos^2 v\cos u\sin u+\sin^2 v\cos u\sin u)$$
$$=(-\cos v\sin^2 u,\ \sin v\sin^2 u,\ \cos u\sin u)$$

$$\|\vec{T}_u\times\vec{T}_v\|=\left[(\cos^2 v\sin^4 u)+(\sin^2 v\sin^4 u)+(\cos^4 v\cos^2 u\sin^2 u+\sin^4 v\cos^2 u\sin^2 u)\right]^{1/2}$$
$$=\left[\sin^4 u(\cos^2 v+\sin^2 v)+\cos^2 u\sin^2 u(\cos^4 v+\sin^4 v)\right]^{1/2}$$
$$=\left[(\cos^2 v+\sin^2 v)\left(\sin^4 u+\cos^2 u\sin^2 u(\cos^2 v+\sin^2 v)\right)\right]^{1/2}$$
$$=\left[\sin^4 u+\cos^2 u\sin^2 u\right]^{1/2}$$
$$=\left[\sin^2 u\left[\sin^2 u+\cos^2 u\right]\right]^{1/2}$$
$$=\sin u$$

*Sphere?

$$\frac{\vec{T}_u\times\vec{T}_v}{\|\vec{T}_u\times\vec{T}_v\|}=\frac{(-\cos v\sin^2 u,\ \sin v\sin^2 u,\ \cos u\sin u)}{\sin u}=(\cos v\sin u,\ \sin v\sin u,\ \cos u)$$

11. $x=\sin v,\ y=u,\ z=\cos v$ for $0\le v\le 2\pi$ and $-1\le u\le 3$ Identify the surface

$\vec{R}(u,v)=(\sin v,\ u,\ \cos v)$

$\vec{T_u}=(0,1,0)$

$\vec{T_v}=(\cos v,\ 0,\ -\sin v)$

Normal vector:

$$\vec{T_u}\times\vec{T_v}=\begin{vmatrix} i & j & k\\ 0 & 1 & 0\\ \cos v & 0 & -\sin v\end{vmatrix}=(-\sin v,\ 0,\ -\cos v)$$

$$\|\vec{T_u}\times\vec{T_v}\|=[(-\sin v)^2+(-\cos v)^2]^{1/2}=1$$

$$\frac{\vec{T_u}\times\vec{T_v}}{\|\vec{T_u}\times\vec{T_v}\|}=(-\sin v,\ 0,\ -\cos v)$$

Cone → increasing in y direction.

12. $x=(2-\cos v)\cos u,\ y=(2-\cos v)\sin u,\ z=\sin v$ for $-\pi\le u\le\pi$, $-\pi\le v\le\pi$. Is this surface regular? Identify the surface.

$\vec{R}(u,v)=((2-\cos v)\cos u,\ (2-\cos v)\sin u,\ \sin v)$

$\vec{T_u}=(-(2-\cos v)\sin u,\ (2-\cos v)\cos u,\ 0)$

$(2-\cos v)\cos u=2\cos u-\cos v\cos u,$

$(2-\cos v)\sin u=2\sin u-\cos v\sin u$

$\vec{T_v}=(\sin v\cos u,\ \sin v\sin u,\ \cos v)$

$$\vec{T_u}\times\vec{T_v}=\begin{vmatrix} i & j & k\\ (\cos v-2)\sin u & (2-\cos v)\cos u & 0\\ \sin v\cos u & \sin v\sin u & \cos v\end{vmatrix}$$

$=((2\cos v-\cos^2 v)\cos u,\ -(\cos^2 v-2\cos v)\sin u,\ (\cos v-2)\sin^2 u\sin v-(2-\cos v)\cos^2 u\sin v))$

$=((2\cos v-\cos^2 v)\cos u,\ -(\cos^2 v-2\cos v)\sin u,\ (\cos v-2)\sin^2 u\sin v+(\cos v-2)\cos^2 u\sin v)$

$=((2\cos v-\cos^2 v)\cos u,\ -(\cos^2 v-2\cos v)\sin u,\ (\cos v-2)[\sin^2 u\sin v+\cos^2 u\sin v])$

$=((2\cos v-\cos^2 v)\cos u,\ -(\cos^2 v-2\cos v)\sin u,\ (\cos v-2)[\sin v(\sin^2 u+\cos^2 u)])$

$=((2\cos v-\cos^2 v)\cos u,\ -(\cos^2 v-2\cos v)\sin u,\ \sin v\cos v-2\sin v)$

$\|\vec{T_u}\times\vec{T_v}\|=[[(2\cos v-\cos^2 v)^2\cos^2 u]+[(\cos^2 v-2\cos v)^2\sin^2 u]+[\sin v\cos v-2\sin v]^2]^{1/2}$

$=[[(4\cos^2 v-4\cos^3 v+\cos^4 v)\cos^2 u]+[(\cos^4 v-4\cos^3 v+4\cos^2 v)\sin^2 u]+[\sin^2 v\cos^2 v-4\sin^2 v\cos v+4\sin^2 v]]^{1/2}$

$=[[\cos^2 v(4-4\cos v+\cos^2 v)]+[\sin^2 v(\cos^2 v-4\cos v+4)]]^{1/2}$

$=[\cos^2 v-4\cos v+4]^{1/2}$

$=[\cos v-2]$

$$\frac{\vec{T_u}\times\vec{T_v}}{\|\vec{T_u}\times\vec{T_v}\|}=\frac{[(2\cos v-\cos^2 v)\cos u,\ -(\cos^2 v-2\cos v)\sin u,\ \sin v\cos v-2\sin v]}{(\cos v-2)}$$

$$=\frac{[-\cos v(\cos v-2)\cos u,\ -\cos v(\cos v-2)\sin u,\ \sin v(\cos v-2)]}{(\cos v-2)}$$

$=[-\cos v\cos u,\ -\cos v\sin u,\ \sin v]$ Torus

15. Find a parametrization of the surface $z=3x^2+8xy$ and use it to find the tangent plane at $x=1,\ y=0,\ z=3$. Compare your answer with that using graphs.

$T(x,y,z)=(x,\ y,\ 3x^2+8xy)$

$\vec{T_x}=(1,\ 0,\ 6x+8y)$

$\vec{T_y}=(0,\ 1,\ 8x)$

Normal vector:

$$\vec{T_x}\times\vec{T_y}=\begin{vmatrix} i & j & k\\ 1 & 0 & 6x+8y\\ 0 & 1 & 8x\end{vmatrix}$$

$=(-6x-8y,\ -8x,\ 1)$

Tangent plane at $(1,0,3)$:

$[-6(x-1)-8(y-0)]+[1(z-3)]=0$

$-6x+6-8y+z-3=0$

$-6x+3-8y+z=0$

$z=6x+8y-3$

16. Find a parametrization of the surface $x^3+3xy+z^2=2,\ z>0$, and use it to find the tangent plane at point $x=1,\ y=\frac{1}{3},\ z=0$. Compare your answer with that using level sets.

Since $z>0$,

$z^2=2-x^3-3xy$

$z=\sqrt{2-x^3-3xy}$ ← let $x=u$, and $y=v$

$T(u,v)=(u,\ v,\ \sqrt{2-u^3-3uv})$

$\vec{T_u}=\left(1,\ 0,\ -\frac{3u^2-3v}{2\sqrt{2-u^3-3uv}}\right)$

$\vec{T_y}=\left(0,\ 1,\ -\frac{3u}{2\sqrt{2-u^3-3uv}}\right)$

$$\vec{T_u}\times\vec{T_v}=\begin{vmatrix} i & j & k\\ 1 & 0 & -\frac{3u^2-3v}{2\sqrt{2-u^3-3uv}}\\ 0 & 1 & -\frac{3u}{2\sqrt{2-u^3-3uv}}\end{vmatrix}$$

$$=\left(\left[\frac{3u^2+3v}{2\sqrt{2-u^3-3uv}}\right],\ \left[\frac{3u}{2\sqrt{2-u^3-3uv}}\right],\ 1\right)$$

$=([3u^2+3v],\ [3u],\ [2\sqrt{2-u^3-3uv}])$

At $(1,\frac{1}{3},0)$,

$\vec{T_u}\times\vec{T_v}=([3+1],\ [3],\ [2\sqrt{2-1-1}])$

$=(4,\ 3,\ 0)$

Tangent plane:

$(x-1,\ y-\frac{1}{3},\ z)\cdot(4,\ 3,\ 0)=0$

$[4(x-1)]+[3(y-\frac{1}{3})]+[0(z)]=0$

$4x-4+3y-1=0$

$4x+3y-5=0$ ⟶ $4x+3y=5$

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19 a) Find the parametrization for the hyperboloid $x^2+y^2-z^2=25$.

cylindrical coordinates: $r^2-z^2=25$

Since $r \geq 0$,

$\hookrightarrow r=\sqrt{25+z^2}$

Parametrization: $\vec{\sigma}:[0,2\pi) \times \mathbb{R} \to \mathbb{R}^3$

$\vec{\sigma}(\theta,z) = (\sqrt{25+z^2}\cos\theta, \; + \sqrt{25+z^2}\sin\theta, \; + z\hat{k})$

is a possible solution

b) Find an expression for the unit normal to the surface.

$\vec{\sigma}_\theta = (-\sqrt{25+z^2}\sin\theta, \; \sqrt{25+z^2}\cos\theta, \; 0)$

$$\vec{\sigma}_z = \left(\frac{z\cos\theta}{\sqrt{25+z^2}}, \; \frac{z\sin\theta}{\sqrt{25+z^2}}, \; 1\right)$$

Normal vector:

$$\vec{\sigma}_\theta \times \vec{\sigma}_z = \begin{vmatrix} i & j & k \\ -\sqrt{25+z^2}\sin\theta & \sqrt{25+z^2}\cos\theta & 0 \\ \frac{z\cos\theta}{\sqrt{25+z^2}} & \frac{z\sin\theta}{\sqrt{25+z^2}} & 1 \end{vmatrix}$$

$$= \left(\sqrt{25+z^2}\cos\theta, \; \sqrt{25+z^2}\sin\theta, \; -z\sin^2\theta - z\cos^2\theta\right)$$

$$= (\sqrt{25+z^2}\cos\theta, \; \sqrt{25+z^2}\sin\theta, \; -z)$$

$$\|\vec{\sigma}_\theta \times \vec{\sigma}_z\| = \sqrt{[\sqrt{25+z^2}\cos\theta]^2 + [\sqrt{25+z^2}\sin\theta]^2 + [-z]^2}$$

$$= \sqrt{[25+z^2\cos^2\theta] + [25+z^2\sin^2\theta) + z^2}$$

$$= \sqrt{25+z^2+z^2}$$

$$= \sqrt{25+2z^2}$$

$$\frac{\vec{\sigma}_\theta \times \vec{\sigma}_z}{\|\vec{\sigma}_\theta \times \vec{\sigma}_z\|} = \left(\frac{\sqrt{25+z^2}\cos\theta}{\sqrt{25+2z^2}}, \; \frac{\sqrt{25+z^2}\sin\theta}{\sqrt{25+2z^2}}, \; \frac{-z}{\sqrt{25+2z^2}}\right)$$

c) Find an equation for the plane tangent to the surface at $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$

The hyperboloid is the level surface at height zero when $f(x,y,z) = x^2+y^2-z^2-25$

At $(x_0,y_0,0)$, the tangent plane:

$f_x(x_0,y_0,0)(x-x_0) + f_y(x_0,y_0,0)(y-y_0) + f_z(x_0,y_0,0)(z-0) = 0$

$2(x_0)(x-x_0) + 2(y_0)(y-y_0) + 2(0)(z-z_0) = 0$

$(x_0)(x-x_0) + (y_0)(y-y_0) + 0 = 0$

$x_0x + x_0^2 + y_0y + y_0^2 + 0 = 0$

Comparing to $f(x,y,z) = x^2+y^2-z^2-25$,

$x_0^2 + y_0^2 - 0 + (x_0x + y_0y) = x^2+y^2-z^2-25$

$\hookrightarrow x_0^2 + y_0^2 - 25 = 0$

$x_0^2 + y_0^2 = 25$

19 d) Show that the lines $(x_0,y_0,0) + t(-y_0, x_0, 5)$ and $(x_0,y_0,0) + t(y_0, -x_0, 5)$ lie in the surface and the tangent plane found in c).

Since $x_0(x_0 \pm ty_0) + y_0(y_0 \mp tx_0) = x_0^2 \pm tx_0y_0 + y_0^2 \mp tx_0y_0$

$= 25$

and

$(x_0 \pm ty_0)^2 + (y_0 \mp tx_0)^2 - (5t)^2 = x_0^2 \pm 2tx_0y_0 + t^2y_0^2 + y_0^2 \mp 2x_0y_0 + t^2x_0^2 - 25t^2$

$= (x_0^2 + y_0)^2 + t^2(x_0^2 + y_0^2 - 25)$

$= 25$

$\hookrightarrow$ the two lines lie on the hyperboloid and the tangent plane.

20. A parametrized surface is described by a differentiable function $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$. According to Chapter 2, the derivative should give a linear approximation that yields a representation of the tangent plane. This exercise demonstrates that this indeed the case.

a). Assuming $\vec{T_u} \times \vec{T_v} \neq 0$, show that the range of the linear transformation $D\Phi(u_0,v_0)$ is in the plane spanned by T_u and T_v. (Here T_u and T_v are evaluated at (u_0,v_0)).

Since $\vec{T_u} \times \vec{T_v} \neq 0$ at (u_0,v_0), the surface is regular at $\Phi(u_0,v_0)$. The tangent plane is determined by $\vec{T_u}$ and $\vec{T_v}$

$$\vec{F} \cdot d\vec{s} = \iint_S (\vec{F} \cdot \vec{n})\,ds = \iint_D \vec{F} \cdot \vec{T_u} \times \vec{T_v}\,du\,dv$$

$$= \iint_D \vec{F} \cdot \frac{\vec{T_u} \times \vec{T_v}}{\|\vec{T_u} \times \vec{T_v}\|} \underbrace{\|\vec{T_u} \times \vec{T_v}\|\,du\,dv}_{ds}$$

b) Show that $\mathbf{w} \perp (T_u \times T_v)$ if and only if $\mathbf{w}$ is in the range of $D\Phi(u_0,v_0)$

($\Rightarrow$) $\vec{w} \perp (\vec{T_u} \times \vec{T_v})$

Since the plane, P, spanned by $\vec{T_u}$ and $\vec{T_v}$ has a codimension 1, and $\vec{T_u} \times \vec{T_v}$ is perpendicular to P, $\vec{w}$ should be in P.

$\therefore e\,\exists\; a,b \in \mathbb{R} \,|\, w = a\vec{T_u} + b\vec{T_v} = \left[D\Phi\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\right]$

($\Leftarrow$) If $\vec{w} \in \text{Image}(D\Phi)$,

$\vec{w} = a\vec{T_u} + b\vec{T_v}$ for some a, b

$\vec{w} \cdot (\vec{T_u} \times \vec{T_v}) = 0$

c) Show that the tangent plane as defined in this section is the same as the "parametrized plane"

$$(u,v) \mapsto (u_0,v_0) + D\Phi(u_0,v_0)\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix}.$$

$\ast$ Linear/Affine approximation of a plane:

$$z = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)$$

$$= f(x_0,y_0) + Df(x_0,y_0)\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

$$D\Phi\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \\ \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0 \end{bmatrix} \begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x}{\partial u}(u_0,v_0)(u-u_0) + \frac{\partial x}{\partial v}(u_0,v_0)(v-v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0)(u-u_0) + \frac{\partial y}{\partial v}(u_0,v_0)(v-v_0) \\ \frac{\partial z}{\partial u}(u_0,v_0)(u-u_0) + \frac{\partial z}{\partial v}(u_0,v_0)(v-v_0 \end{bmatrix}$$

$$= \vec{T_u}(u-u_0) + \vec{T_v}(v-v_0)$$

Since $\Phi(u_0, v_0) = [x(u_0, v_0),\ y(u_0, v_0),\ z(u_0, v_0)]$,

$$\Phi(u_0, v_0) + \vec{D}\Phi(u_0, v_0)\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix} = \underbrace{\Phi(u_0, v_0) + \vec{T}_u(u-u_0) + \vec{T}_v(v-v_0)}_{\text{linear approximation as in chapt 2.}}$$