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HW #5

Excellent! 2/2

7.4

① $\vec{\phi}: D \to S \subset \mathbb{R}^3$ $\quad x = \cos\theta \sin\phi \quad y = \sin\theta\sin\phi \quad z = \cos\phi$

$0 \le \theta \le 2\pi$ $\quad \vec{T}_\theta = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$

$0 \le \phi \le \pi$ $\quad \vec{T}_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$

$$\vec{T}_\theta \times \vec{T}_\phi = (-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi)$$

$$\|\vec{T}_\theta \times \vec{T}_\phi\| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi} = \sin\phi$$

$$A(S) = \iint_D \|\vec{T}_\theta \times \vec{T}_\phi\| \, d\theta\, d\phi = \int_0^\pi\int_0^{2\pi} \sin\phi \, d\theta\, d\phi = 2\pi\int_0^\pi \sin\phi\, d\phi$$

$$= 2\pi\left[-\cos\phi\right]_0^\pi = 2\pi\left[-(-1)-(-1)\right] = 2\pi[2] = \boxed{4\pi}$$

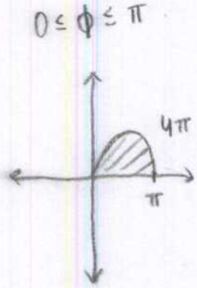
② 1) $-\pi/2 \le \phi \le \pi/2$

$$2\pi\int_{-\pi/2}^{\pi/2} \sin\phi\, d\phi = 2\pi\left[-\cos\phi\right]_{-\pi/2}^{\pi/2} = 2\pi[0-0] = 0$$

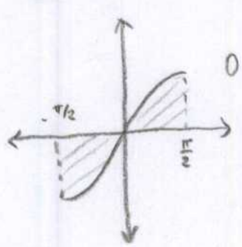
2) $0 \le \phi \le 2\pi$

$$2\pi\int_0^{2\pi}\sin\phi\, d\phi = 2\pi\left[-\cos\phi\right]_0^{2\pi} = 2\pi\left[-1-(-1)\right] = 2\pi[0] = 0$$

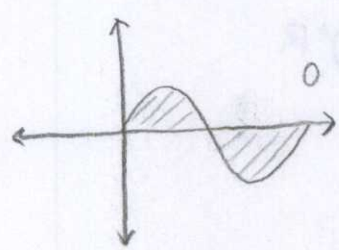
$0 \le \phi \le \pi$



$-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$



$0 \le \phi \le 2\pi$



equal area on positive + negative sides which add up to 0

(4) $\vec{\Phi}: D \to \mathbb{R}^3$ $\quad x = (R+\cos\phi)\cos\theta \quad y = (R+\cos\phi)\sin\theta \quad z = \sin\phi$

$0 \le \theta \le 2\pi$

$0 \le \phi \le 2\pi$

Formula 3

$$\vec{T}_\theta = \left(-(R+\cos\phi)\sin\theta,\ (R+\cos\phi)\cos\theta,\ 0\right)$$

$$\vec{T}_\phi = \left(-\sin\phi\cos\theta,\ -\sin\phi\sin\theta,\ \cos\phi\right)$$

$$\vec{T}_\theta \times \vec{T}_\phi = \left((R+\cos\phi)\cos\theta\cos\phi,\ (R+\cos\phi)\sin\theta\cos\phi,\ (R+\cos\phi)\sin\phi\right)$$

$$\|\vec{T}_\theta \times \vec{T}_\phi\| = \sqrt{(R+\cos\phi)^2\cos^2\theta\cos^2\phi + (R+\cos\phi)^2\sin^2\theta\cos^2\phi + (R+\cos\phi)^2\sin^2\phi} = R+\cos\phi$$

$$A(S) = \int_0^{2\pi}\int_0^{2\pi} (R+\cos\phi)\,d\theta\,d\phi = 2\pi\int_0^{2\pi}(R+\cos\phi)\,d\phi = 2\pi\left[R\phi+\sin\phi\right]_0^{2\pi}$$

$$= 2\pi(R2\pi) = \boxed{(2\pi)^2 R}$$

Formula 6

$$A = 2\pi\int_a^b \left(|x|\sqrt{1+f'(x)^2}\right)dx$$

In Cartesian coordinates $(y-R)^2+z^2=1$ when revolved around the z-axis, makes a torus

Therefore, $z = \sqrt{1-(y-R)^2}$

$y = R \pm \sqrt{1-z^2}$

$$A(S) = 2\pi\int \left(R+\cancel{\sqrt{1-z^2}}\right)\sqrt{1+\left(\frac{dz}{dy}\right)^2}\,dy + 2\pi\int\left(R-\cancel{\sqrt{1-z^2}}\right)\sqrt{1+\left(\frac{dz}{dy}\right)^2}\,dy$$

$$= 2\pi\int 2R\sqrt{1+\left(\frac{dz}{dy}\right)^2}\,dy = 4\pi R\int\sqrt{1+\left(\frac{dz}{dy}\right)^2}\,dy = 4\pi R\pi$$

$$= (2\pi)^2 R$$

(5) $\phi(u,v) = (e^u \cos v, e^u \sin v, v)$ $\qquad \vec{T}_u = (e^u \cos v, e^u \sin v, 0)$

$0 \le u \le 1 \quad 0 \le v \le \pi$ $\qquad \vec{T}_v = (-e^u \sin v, e^u \cos v, 1)$ $\qquad {}^*\cos^2 v + \sin^2 v = 1$

a) $\vec{T}_u \times \vec{T}_v = (e^u \sin v, -e^u \cos v, e^u \cos^2 v + e^u \sin^2 v) = (e^u \sin v, -e^u \cos v, e^u)$

b) $(u,v) = (0, \pi/2)$

$\vec{T}_u \times \vec{T}_v$ at $(0, \pi/2) = (1, 0, 1)$

$\phi(u,v)$ at $(0,\pi/2)$: $x = 0$, $y = 1$, $z = \pi/2$

$1(x-0) + 0(y-1) + 1(z - \pi/2) = 0$

$x + z - \pi/2 = 0 \quad \Rightarrow \quad \boxed{x + z = \pi/2}$

c) $\|\vec{T}_u \times \vec{T}_v\| = \sqrt{e^{2u}\sin^2 v + e^{2u}\cos^2 v + e^{2u}} = \sqrt{2e^{2u}} = \sqrt{2}\,e^u$

$$\int_0^1 \int_0^\pi \sqrt{2}\,e^u \,dv\,du = \sqrt{2}\,\pi \int_0^1 e^u \,du = \sqrt{2}\,\pi \left[e^u\right]_0^1 = \boxed{\sqrt{2}\,\pi(e-1)}$$

(6) $z = xy$; $x^2 + y^2 \le 2$

$\phi(u,v) = (u, v, uv) \qquad u^2 + v^2 \le 2$

$\vec{T}_u = (1, 0, v) \qquad \vec{T}_v = (0, 1, u)$

$\vec{T}_u \times \vec{T}_v = (-v, -u, 1) \qquad \|\vec{T}_u \times \vec{T}_v\| = \sqrt{v^2 + u^2 + 1}$

$A(S) = \iint_D \sqrt{v^2 + u^2 + 1}\,du\,dv$

POLAR COORDINATES

$u^2 + v^2 = r^2$

$u = r\cos\theta \qquad 0 \le r \le \sqrt{2}$

$v = r\sin\theta \qquad 0 \le \theta \le 2\pi$

Jac. $= r$

$$\int_0^{\sqrt{2}} \int_0^{2\pi} \sqrt{1+r^2}\; r \,d\theta\,dr = 2\pi \int_0^{\sqrt{2}} \sqrt{1+r^2}\; r\,dr$$

$$= \pi \int_1^3 \sqrt{t}\,dt = \pi \frac{2}{3}\left[t^{3/2}\right]_1^3$$

$t = 1 + r^2$

$dt = 2r\,dr$

$\frac{dt}{2} = r\,dr$

$$= \boxed{\frac{2}{3}\pi\left(3\sqrt{3} - 1\right)}$$

(9) $\phi(u,v) = (u-v, u+v, uv)$ $\qquad \phi(D) = A(S)$

$\vec{T}_u = (1, 1, v) \qquad \vec{T}_u \times \vec{T}_v = (u-v, -(u+v), 2)$

$\vec{T}_v = (-1, 1, u) \qquad \|\vec{T}_u \times \vec{T}_v\| = \sqrt{(u-v)^2 + (u+v)^2 + 4} = \sqrt{2(u^2+v^2)+4}$

$(u^2 - \cancel{2uv} + v^2) + (u^2 + \cancel{2uv} + v^2)$

UNIT DISC
POLAR COORDINATES

$u^2+v^2 \le 1$

$u^2+v^2 = r^2 \qquad 0 \le r \le 1$

$u = r\cos\theta \qquad 0 \le \theta \le 2\pi$

$v = r\sin\theta$

$J = r$

$$A(S) = \int_0^1 \int_0^{2\pi} \sqrt{4+2r^2}\, r\, d\theta\, dr = 2\pi \int_0^1 \sqrt{4+2r^2}\, r\, dr$$

$$= \frac{\pi}{2} \int_4^6 \sqrt{t}\, dt = \frac{2}{3} \frac{\pi}{2} \left[t^{3/2} \right]_4^6$$

$$= \boxed{\frac{\pi}{3}\left(6\sqrt{6} - 8\right)}$$

$t = 4 + 2r^2$

$dt = 4r\, dr \qquad \frac{dt}{4} = r\, dr$

(17) $x+y+z=1$; $x^2+2y^2 \le 1$

$z = 1-x-y$

$\phi(u,v) = (u, v, 1-u-v)$

$\vec{T}_u = (1, 0, -1) \qquad \vec{T}_u \times \vec{T}_v = (1, 1, 1)$

$\vec{T}_v = (0, 1, -1) \qquad \|\vec{T}_u \times \vec{T}_v\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

ELLIPSE

$u^2+2v^2 \le 1$

$-1 \le u \le 1$

$-\sqrt{\frac{1-u^2}{2}} \le v \le \sqrt{\frac{1-u^2}{2}}$

$$A(S) = \int_{-1}^{1} \int_{-\sqrt{\frac{1-u^2}{2}}}^{\sqrt{\frac{1-u^2}{2}}} \sqrt{3}\, dv\, du$$

$$= \sqrt{3} \int_{-1}^{1} \cancel{2}\left(\frac{\sqrt{1-u^2}}{\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}} du = \sqrt{3} \int_{-1}^{1} \sqrt{2}\sqrt{1-u^2}\, du$$

$u = \sin t$

$du = \cos t\, dt$

$t = \sin^{-1}(u)$

$$= \sqrt{6} \int_{-1}^{1} \sqrt{1-u^2}\, du = \sqrt{6} \int_{-1}^{1} \sqrt{1-\sin^2 t}\, \cos t\, dt = \sqrt{6} \int_{-1}^{1} \cos^2 t\, dt$$

Reduction formula $\int \cos^n(t)\, dt = \frac{n-1}{n} \int \cos^{n-2}(t)\, dt + \frac{\cos^{n-1}(t)\sin t}{n} \qquad n=2$

$*1-\sin^2 t = \cos^2 t$

$$\frac{1}{2}\int 1\, dt + \frac{\cos(t)\sin(t)}{2} = \frac{1}{2}t + \frac{\cos(t)\sin(t)}{2}$$

$$= \sqrt{6}\left[\frac{1}{2}\arcsin(u) + \frac{u\sqrt{1-u^2}}{2}\right]_{-1}^{1} = \sqrt{6}\left(\left(\frac{\pi/2}{2}+0\right) - \left(-\frac{\pi/2}{2}+0\right)\right) = \sqrt{6}\,\frac{2\pi}{4}$$

$$= \boxed{\sqrt{6}\,\frac{\pi}{2}}$$

(22) $z = f(x,y)$, $(x,y) \in D \subset \mathbb{R}^3$ $\sim$ $(x,y,z) \in \mathbb{R}^3$ $F(x,y,z) = 0$

$\vec{T}_x = \left(1, 0, \frac{\partial f}{\partial x}\right)$ $\vec{T}_x \times \vec{T}_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$

$\vec{T}_y = \left(0, 1, \frac{\partial f}{\partial y}\right)$ $\|\vec{T}_x \times \vec{T}_y\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$

$$A(S) = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \; dA$$

$F(x,y,z) = f(x,y) - z = 0$

therefore, $\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x}$ $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + (-1)^2 = \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2$

$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y}$ $\Rightarrow$ $= \|\nabla F\|^2$

$\frac{\partial F}{\partial z} = -1$

$$\boxed{\sim A(S) = \iint_D \|\nabla F\| \, dA}$$

7.5

(2) $f(x,y,z) = z + 6$ $f = z + 6 = v + 6$ $\vec{T}_u = (1, 0, 0)$

$\phi(u,v) = \left(u, \frac{v}{3}, v\right)$ $f(\phi(u,v)) = v + 6$ $\vec{T}_v = \left(0, \frac{1}{3}, 1\right)$

$0 \le u \le 2$ $0 \le v \le 3$ $\vec{T}_u \times \vec{T}_v = \left(0, -1, \frac{1}{3}\right)$

$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{1 + 1/9} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}$

$$\iint_D f(\phi(u,v)) \|\vec{T}_u \times \vec{T}_v\| \, du\,dv$$

$$= \int_0^3 \int_0^2 (v+6) \frac{\sqrt{10}}{3} \, du\,dv = \frac{2\sqrt{10}}{3} \int_0^3 (v+6)\, dv$$

$$= \frac{2\sqrt{10}}{3} \left[\frac{1}{2}v^2 + 6v\right]_0^3 = \frac{2\sqrt{10}}{3}\left[\left(\frac{9}{2} + 18\right) - (0+0)\right] = \frac{2\sqrt{10}}{3}\left(\frac{45}{2}\right) = \frac{90\sqrt{10}}{6}$$

$$= \boxed{15\sqrt{10}}$$

(3) $\iint_S (3x-2y+z)\,dS$ plane: $2x+3y+z=6 \Rightarrow z=6-2x-3y$

$\phi(u,v)=(u,v,6-2u-3v)$

$\vec{T}_u=(1,0,-2)$

$\vec{T}_v=(0,1,-3)$

$\vec{T}_u\times\vec{T}_v=(-2,3,1)$

$\|\vec{T}_u\times\vec{T}_v\|=\sqrt{4+9+1}=\sqrt{14}$

$f(\phi(u,v))=3u-2v+6-2u-3v$
$=6+u-5v$

$0\le u\le 3$

$0\le v\le \frac{6-2u}{3}$

$$=\int_0^3\int_0^{\frac{6-2u}{3}}(6+u-5v)\sqrt{14}\,dv\,du$$

$$=\sqrt{14}\int_0^3\left[6v+uv-\frac{5}{2}v^2\right]_0^{\frac{6-2u}{3}}du$$

$$=\sqrt{14}\int_0^3 2+\frac{14}{3}u-\frac{16}{9}u^2\,du$$

$$=\sqrt{14}\left(2u+\frac{14}{6}u^2-\frac{16}{27}u^3\right)\Big|_0^3=\boxed{11\sqrt{14}}$$

(4) $\iint_S (x+z)\,dS$ S is part of cylinder $y^2+z^2=4$ $x\in[0,5]$

$\phi=(x,2\cos\theta,2\sin\theta)$ $0\le x\le 5$, $0\le\theta\le 2\pi$

$\vec{T}_x=(1,0,0)$

$\vec{T}_\theta=(0,-2\sin\theta,2\cos\theta)$

$\vec{T}_x\times\vec{T}_\theta=(0,-2\cos\theta,-2\sin\theta)$

$\|\vec{T}_x\times\vec{T}_\theta\|=\sqrt{4\cos^2\theta+4\sin^2\theta}=\sqrt{4}=2$

$f(\phi(u,v))=x+2\sin\theta$

$$\int_0^{2\pi}\int_0^5(x+2\sin\theta)2\,dx\,d\theta$$

$$=2\int_0^{2\pi}\left(\frac{1}{2}x^2+2\sin\theta x\right)\Big|_0^5 d\theta=2\int_0^{2\pi}\left(\frac{25}{2}+10\sin\theta\right)d\theta=2\left[\frac{25}{2}\theta-10\cos\theta\right]_0^{2\pi}$$

$$=2\left[25\pi-\cancel{10}+\cancel{10}\right]=\boxed{50\pi}$$

(6) $\iint_S (x^2z+y^2z)\,dS$ S is part of plane: $z=4+x+y$ inside cylinder $x^2+y^2=4$

$\phi(u,v)=(u,v,4+u+v)$

$\vec{T}_u=(1,0,1)$ $\vec{T}_u\times\vec{T}_v=(1,-1,1)$

$\vec{T}_v=(0,1,1)$ $\|\vec{T}_u\times\vec{T}_v\|=\sqrt{3}$

CYLINDER

$x^2+y^2=4$

$x=r\cos\theta$ $0\le r\le 2$

$y=r\sin\theta$ $0\le\theta\le 2\pi$

$z=4$

$J=r$

$$\int_0^{2\pi}\int_0^2\left(r^2\cos^2\theta(4)+r^2\sin^2\theta(4)\right)r\sqrt{3}\,dr\,d\theta$$

$$=\sqrt{3}\int_0^{2\pi}\int_0^2 4r^3\,dr\,d\theta=\sqrt{3}\int_0^{2\pi}\frac{4}{4}r^4\Big]_0^2 d\theta=\sqrt{3}\int_0^{2\pi}16\,d\theta=\sqrt{3}\cdot16\cdot2\pi$$

$$=\boxed{32\sqrt{3}\pi}$$

7) $\iint_S xy\,dS$ S: surface of tetrahedron with sides $z=0$, $y=0$, $x+z=1$, $x=y$

vertices: $(1,0,0)$, $(1,1,0)$, $(0,0,0)$, $(0,0,1)$

$$=\iint xy\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

1) Vertices $(1,0,0)$, $(1,1,0)$, $(0,0,1)$

$x+z=1$ $0\le y\le x$

$z=1-x$ $0\le x\le 1$

$$\iint xy\,dS=\int_0^1\int_0^x xy\sqrt{1+(-1)^2+0^2}\,dy\,dx=\int_0^1\int_0^x xy\sqrt{2}\,dy\,dx=\sqrt{2}\int_0^1\left[x\tfrac{1}{2}y^2\right]_0^x dx$$

$$=\sqrt{2}\int_0^1 \tfrac{1}{2}x^3\,dx=\sqrt{2}\left[\tfrac{1}{8}x^4\right]_0^1=\tfrac{1}{8}\sqrt{2}$$

2) Vertices $(1,0,0)$, $(1,1,0)$, $(0,0,0)$

$z=0$ $0\le y\le x$ $0\le x\le 1$

$$\iint xy\,dS=\int_0^1\int_0^x xy\sqrt{1+0^2+0^2}\,dy\,dx=\int_0^1\int_0^x xy\,dy\,dx=\int_0^1\left[x\tfrac{1}{2}y^2\right]_0^x dx$$

$$=\tfrac{1}{2}\int_0^1 x^3\,dx=\tfrac{1}{2}\tfrac{1}{4}x^4\Big]_0^1=\tfrac{1}{8}$$

3) Vertices $(0,0,0)$, $(1,1,0)$, $(0,0,1)$

$y=x$ $0\le z\le 1-x$ $0\le x\le 1$

$$\iint xy\,dS=\iint xy\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA=\int_0^1\int_0^{1-x} x^2\sqrt{1+(-1)^2+0^2}\,dz\,dx$$

$$=\sqrt{2}\int_0^1\int_0^{1-x} x^2\,dz\,dx=\sqrt{2}\int_0^1\left[x^2 z\right]_0^{1-x}dx=\sqrt{2}\int_0^1 x^2(1-x)\,dx$$

$$=\sqrt{2}\int_0^1(x^2-x^3)\,dx=\sqrt{2}\left[\tfrac{1}{3}x^3-\tfrac{1}{4}x^4\right]_0^1=\sqrt{2}\left(\tfrac{1}{3}-\tfrac{1}{4}\right)=\tfrac{1}{12}\sqrt{2}$$

4) Vertices $(0,0,0)$, $(1,0,0)$, $(0,0,1)$

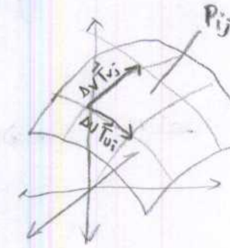
$y=0$ $=0$

$$\tfrac{1}{8}\sqrt{2}+\tfrac{1}{8}+\tfrac{1}{12}\sqrt{2}+0=\frac{5\sqrt{2}}{24}+\frac{3}{24}=\boxed{\frac{3+5\sqrt{2}}{24}}$$

(18) a) $\dfrac{1}{A(S)} \iint_S f(x,y,z)\, dS$

According to textbook, the integral of f over S by using Riemann Sums is:

$\lim_{n\to\infty} S_n = \iint_S f\, dS$ and $S_n = \sum_{i=1}^{n}\sum_{i=1}^{n} f(\vec{\Phi}(u_i,v_j)) \| \vec{T}_{u_i} \times \vec{T}_{v_j} \| \Delta u \,\Delta v$



where we have $\| \vec{T}_{u_i} \times \vec{T}_{v_j} \| \Delta u\, \Delta v = A(P_{ij}) = \| \Delta u\, \vec{T}_{u_i} \times \Delta v\, \vec{T}_{v_j} \|$

Then, $\sum_{i=1}^{n}\sum_{i=1}^{n} A(P_{ij}) = \sum_{i=1}^{n}\sum_{i=1}^{n} \| \vec{T}_{u_i} \times \vec{T}_{v_j} \| \Delta u\, \Delta v = \iint_D \| \vec{T}_u \times \vec{T}_v \| \, du\, dv = A(S)$

Therefore, $\lim_{n\to\infty} \dfrac{\sum_{i=1}^{n}\sum_{i=1}^{n} f(\vec{\Phi}(u_i,v_j)) \| \vec{T}_{u_i} \times \vec{T}_{v_j} \| \Delta u\, \Delta v}{\sum_{i=1}^{n}\sum_{i=1}^{n} \| \vec{T}_{u_i} \times \vec{T}_{v_j} \| \Delta u\, \Delta v} = \dfrac{\sum_{i=1}^{n}\sum_{i=1}^{n} f(\vec{\Phi}(u_i,v_j)) \| \vec{T}_{u_i} \times \vec{T}_{v_j} \| \Delta u\, \Delta v}{A(S)}$

You took limit here and not on the numerator. You want to take limit on both simultaneously.

~~Thus~~, $= \dfrac{\iint_S f(x,y,z)\, dS}{A(S)}$

b) According to textbook, example 3 shows that $\iint_S z^2\, dS = \dfrac{4\pi}{3}$

Now, $A(S) = \iint_D \| \vec{T}_u \times \vec{T}_v \| \, du\, dv = \int_0^{\pi}\int_0^{2\pi} \sin\phi \, d\theta\, d\phi = 2\pi \int_0^{\pi} \sin\phi\, d\phi$

$\| \vec{T}_u \times \vec{T}_v \| = \sin\phi$ $\qquad = 2\pi \left[-\cos\phi\right]_0^{\pi} = 4\pi$

Thus, $\dfrac{\iint_S f(x,y,z)\, dS}{A(S)} = \dfrac{\cancel{4\pi}}{3} \cdot \dfrac{1}{\cancel{4\pi}} = \boxed{\dfrac{1}{3}}$

c) In example 4,

$\iint_S x\, dS = \sqrt{3} \int_0^1 \int_0^{1-x} x\, dy\, dx = \dfrac{\sqrt{3}}{6}$

Then $M(S) = \iint_S m(x,y,z)\, dS = \sqrt{3} \int_0^1 \int_0^{1-x} dy\, dx = \sqrt{3} \int_0^1 (1-x)\, dx$

$= \sqrt{3} \left[x - \frac{1}{2}x^2 \right]_0^1 = \sqrt{3} \left(1 - \frac{1}{2}\right) = \dfrac{\sqrt{3}}{2}$

$\dfrac{\iint_S x\, dS}{M(S)} = \dfrac{\cancel{\sqrt{3}}}{6} \cdot \dfrac{2}{\cancel{\sqrt{3}}} = \dfrac{1}{3}$

Since it is $(1,0,0), (0,1,0), (0,0,1)$ then the center of gravity for the triangle is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

(23) $\vec{\phi}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$

$x = x(u,v) \quad y = y(u,v) \quad z = z(u,v)$

a) $\frac{\partial \vec{\phi}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \vec{T}_u \qquad \frac{\partial \vec{\phi}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \vec{T}_v$

$E = \left\|\frac{\partial \vec{\phi}}{\partial u}\right\|^2 \qquad F = \frac{\partial \vec{\phi}}{\partial u} \cdot \frac{\partial \vec{\phi}}{\partial v} \qquad G = \left\|\frac{\partial \vec{\phi}}{\partial v}\right\|^2$

$\sim \sqrt{EG - F^2} = \sqrt{\left\|\frac{\partial \phi}{\partial u}\right\|^2 \left\|\frac{\partial \phi}{\partial v}\right\|^2 - \left(\frac{\partial \phi}{\partial u} \cdot \frac{\partial \phi}{\partial v}\right)^2}$

Let $\frac{\partial \phi}{\partial u} \cdot \frac{\partial \phi}{\partial v} = \left\|\frac{\partial \phi}{\partial u}\right\| \left\|\frac{\partial \phi}{\partial v}\right\| \cos\theta$

Then, $\sqrt{EG-F^2} = \sqrt{\left(\left\|\frac{\partial \phi}{\partial u}\right\| \left\|\frac{\partial \phi}{\partial v}\right\|\right)^2 (1-\cos^2\theta)} = \sqrt{\left(\left\|\frac{\partial \phi}{\partial u}\right\| \left\|\frac{\partial \phi}{\partial v}\right\|\right)^2 \sin^2\theta} = \left\|\frac{\partial \phi}{\partial u}\right\| \left\|\frac{\partial \phi}{\partial v}\right\| \sin\theta$

$= \|\vec{T}_u \times \vec{T}_v\|$

Thus, $\boxed{\sqrt{EG-F^2} = \|\vec{T}_u \times \vec{T}_v\|}$

Since $\sqrt{EG-F^2} = \|\vec{T}_u \times \vec{T}_v\|$

$A(S) = \iint_D \|\vec{T}_u \times \vec{T}_v\| \, du\,dv \quad \Leftrightarrow \quad A(S) = \iint_D \sqrt{EF - G^2} \, du\,dv$

b) if $\frac{\partial \vec{\phi}}{\partial u}$ & $\frac{\partial \vec{\phi}}{\partial v}$ are orthogonal, then F must $= 0$

then $\sqrt{EG-F^2} = \sqrt{EG} = \left\|\frac{\partial \phi}{\partial u}\right\| \left\|\frac{\partial \phi}{\partial v}\right\| \quad \sim \boxed{A(S) = \iint_D \left\|\frac{\partial \phi}{\partial u}\right\| \left\|\frac{\partial \phi}{\partial v}\right\| du\,dv}$

c) $\vec{\phi}(\theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$

$\vec{T}_\theta = (-r\sin\theta\sin\phi, r\cos\theta\sin\phi, 0) \qquad \vec{T}_\phi = (r\cos\theta\cos\phi, r\sin\theta\cos\phi, -r\sin\phi)$

$E = \left\|\frac{\partial \vec{\phi}}{\partial \theta}\right\|^2 = r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta\sin^2\phi = r^2\sin^2\phi$

$G = \left\|\frac{\partial \vec{\phi}}{\partial \phi}\right\|^2 = r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta\cos^2\phi + r^2\sin^2\phi = r^2$

$F = \left(\frac{\partial \vec{\phi}}{\partial \theta} \cdot \frac{\partial \vec{\phi}}{\partial \phi}\right) = -r^2\sin\theta\sin\phi\cos\theta\cos\phi + r^2\cos\theta\sin\phi\sin\theta\cos\phi + 0$

$= \sin\theta\sin\phi\cos\theta\cos\phi(r^2 - r^2) = 0$

$\hookrightarrow$

$$\sqrt{EG-F^2} = \sqrt{r^4 \sin^2\phi} = r^2 \sin\phi \qquad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{matrix}$$

$$A(S) = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\phi \, d\phi \, d\theta = 2\pi \int_0^{\pi} r^2 \sin\phi \, d\phi = 2\pi \left[-r^2\cos\phi\right]_0^{\pi}$$

$$= 2\pi r^2 \left[-\cos\phi\right]_0^{\pi} = 2\pi r^2 \left[(-(-1)-(-1)\right] = 2\pi r^2 (2) = \boxed{4\pi r^2}$$

(24) $\vec{\Phi} : D \to \mathbb{R}^3$

$$J(\vec{\Phi}) = \frac{1}{2}\iint_D \left(\left\| \frac{\partial \vec{\Phi}}{\partial u} \right\|^2 + \left\| \frac{\partial \vec{\Phi}}{\partial v} \right\|^2 \right) du\,dv \qquad A(\vec{\Phi}) \le J(\vec{\Phi})$$

$$\vec{T}_u = \frac{\partial \Phi}{\partial u} \quad \vec{T}_v = \frac{\partial \Phi}{\partial v}$$

a) $A(\vec{\Phi}) = \iint_D \| \vec{T}_u \times \vec{T}_v \| \, du\,dv$

$$= \iint_D \| \vec{T}_u \| \, \| \vec{T}_v \| \sin\theta \, du\,dv$$

if $(\|\vec{T}_u\| - \|\vec{T}_v\|)^2 \ge 0$, then $\|\vec{T}_u\|^2 + \|\vec{T}_v\|^2 \ge 2\|\vec{T}_u\|\,\|\vec{T}_v\|$

factoring

$$\Rightarrow \frac{1}{2}\left[\|\vec{T}_u\|^2 + \|\vec{T}_v\|^2\right] \ge \|\vec{T}_u\|\,\|\vec{T}_v\| \ge \|\vec{T}_u\|\,\|\vec{T}_v\| \sin\theta$$

therefore $\iint_D \frac{1}{2}\left(\|\vec{T}_u\|^2 + \|\vec{T}_v\|^2\right) du\,dv \ge \iint_D \|\vec{T}_u\|\,\|\vec{T}_v\| \sin\theta \, du\,dv$

implying $\quad J(\vec{\Phi}) \ge A(\vec{\Phi})$

thus when $\|\vec{T}_u\| = \|\vec{T}_v\|$ and $\sin\theta = 1$

then a) $\|\vec{T}_u\|^2 = \|\vec{T}_v\|^2$ & b) $\vec{T}_u \cdot \vec{T}_v = 0$

hold true