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Section 7.4

Excellent!

1. $\Phi: D \to \sigma \in \mathbb{R}^3$. D is the rectangle $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. $\Phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$

$T_\theta = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0 \rangle$

$T_\phi = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi \rangle$

$T_\theta \times T_\phi = \langle -\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi \rangle$

$\|T_\theta \times T_\phi\| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$

$= \sin\phi$

$A(S) = \iint_D \|T_\theta \times T_\phi\| \, d\theta d\phi = \int_0^\pi \int_0^{2\pi} \sin\phi \, d\theta d\phi$

$= 2\pi \int_0^\pi \sin\phi \, d\phi$

$= 4\pi$

2. $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ and $0 < \phi < 2\pi$.

$A(S) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \sin\phi \, d\phi = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\phi d\phi = 2\pi (-\cos\phi) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$

$A(S) = \int_0^{2\pi} \int_0^{2\pi} \sin\phi d\phi = 2\pi \int_0^{2\pi} \sin\phi d\phi = 2\pi (-\cos\phi) \Big|_0^{2\pi} = 0$

4. $\Phi: D \to \mathbb{R}^3$, $\Phi \langle (R+\cos\phi)\cos\theta, (R+\cos\phi)\sin\theta, \sin\phi \rangle$, $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$, $R > 1$, $A(T) = (2\pi)^2 R$

① $T_\theta = \langle -\sin\theta (R+\cos\phi), \cos\theta (R+\cos\phi), 0 \rangle$

$T_\phi = \langle -\sin\phi\cos\theta, -\sin\phi\sin\theta, \cos\phi \rangle$

$T_\theta \times T_\phi = \langle \cos\phi\cos\theta(R+\cos\phi), \cos\phi\sin\theta(R+\cos\phi), \sin\phi(R+\cos\phi) \rangle$

$\|T_\theta \times T_\phi\| = \sqrt{\cos^2\phi\cos^2\theta(R+\cos\phi)^2 + \cos^2\phi\sin^2\theta(R+\cos\phi)^2 + \sin^2\phi(R+\cos\phi)^2}$

$= R + \cos\phi$

$A(S) = \int_0^{2\pi} \int_0^{2\pi} (R+\cos\phi) \, d\theta d\phi = 2\pi \cdot R \cdot 2\pi = (2\pi)^2 R$

② $(y-R)^2+z^2=1$

$z=\sqrt{1-(y-R)^2}$

Excellent!

$y_1=R+\sqrt{1-z^2}$, $y_2=R-\sqrt{1-z^2}$

$$A(S)=2\pi\int |y_1|\sqrt{1+(\frac{dz}{dy})^2}\,dy+\int |y_2|\sqrt{1+(\frac{dz}{dy})^2}\,dy$$

$$=2\pi\int (y_1+y_2)\sqrt{1+(\frac{dz}{dy})^2}\,dy$$

$$=2R(2\pi)\int\sqrt{1+(\frac{dz}{dy})^2}\,dy$$

$$=2R\cdot 2\pi\cdot\pi$$

$$=(2\pi)^2R$$

6. $\Phi(u,v)=(e^u\cos v, e^u\sin v, v)$, $D=[0,1]\times[0,\pi]$

a) $T_u=(e^u\cos v, e^u\sin v, 0)$

$T_v=(-e^u\sin v, e^u\cos v, 1)$

$T_u\times T_v=(e^u\sin v, -e^u\cos v, e^u)$

b) $T_u\times T_v$ at $(0,\frac{\pi}{2})=(1,0,1)$

$x=0$, $y=1$, $z=\frac{\pi}{2}$

Tangent plane : $1(x)+1(z-\frac{\pi}{2})=0$

$x+z=\frac{\pi}{2}$

c) $\|T_u\times T_v\|=\sqrt{e^{2u}\sin^2v+e^{2u}\cos^2v+e^{2u}}$

$=e^u\sqrt{2}$

$$A(S)=\int_0^\pi\int_0^1 e^u\sqrt{2}\,du\,dv$$

$$=(e-1)\pi\sqrt{2}$$

6. $z = xy$ and $x^2+y^2 \le 2$

$\Phi(u,v) = (u, v, uv)$

$T_u = (1, 0, v)$

$T_v = (0, 1, u)$

$T_u \times T_v = (-v, -u, 1)$

$\|T_u \times T_v\| = \sqrt{v^2+u^2+1}$

in polar co-ordinates:

$u^2+v^2 = e^2$

$u = e\cos\theta \Rightarrow 0 \le e \le \sqrt{2}$

$v = e\sin\theta \rightarrow 0 \le \theta \le 2\pi$

$A(s) = \iint_D \sqrt{v^2+u^2+1}\, du\,dv$

$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+e^2}\, e\,de\,d\theta$

$= 2\pi \cdot \frac{1}{2} \frac{(1+e^2)^{3/2}}{\frac{3}{2}} \Big|_0^{\sqrt{2}}$

$= \frac{2\pi}{3} (1+e^2)^{3/2} \Big|_0^{\sqrt{2}}$

$= \frac{2\pi}{3} (3\sqrt{3} - 1)$

9. $\Phi(u,v) = (u-v, u+v, uv)$

$T_u = (1, 1, v)$

$T_v = (-1, 1, u)$

$T_u \times T_v = (u-v, u+v, 2)$

$\|T_u \times T_v\| = \sqrt{(u-v)^2 + (u+v)^2 + 4} = \sqrt{4+2(u^2+v^2)}$

$\because$ D is unit disc

$\therefore u^2+v^2 \le 1$, $u^2+v^2 = e^2$

$u = e^{\cos\theta} \rightarrow 0 \le e \le 1$

$v = e^{\sin\theta} \rightarrow 0 \le \theta \le 2\pi$

$A(s) = \int_0^{2\pi}\int_0^1 \sqrt{4+2e^2}\, e\,de\,d\theta$

$= \frac{\pi}{3} (6^{3/2} - 4^{3/2})$

17. $x+y+z=1,\ x^2+2y^2\le 1$

$z=1-x-y$

$T_x \times T_y = (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)$

$= (1,1,1)$

$\|T_x \times T_y\| = \sqrt{3}$

$$A(S) = \iint_D \sqrt{3}\, dx dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{\frac{1-x^2}{2}}}^{\sqrt{\frac{1-x^2}{2}}} \sqrt{3}\, dx dy$$

$$= \sqrt{3}\int_{-1}^{1} 2\sqrt{\frac{1-x^2}{2}}\, dx$$

$$= \sqrt{6}\int_{-1}^{1}\sqrt{1-x^2}\, dx$$

$$= \sqrt{6}\left(\frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}(x)}{2}\right)\Big|_{-1}^{1}$$

$$= \frac{\pi}{2}\sqrt{6}$$

22. $z=f(x,y)$ where $(x,y)\in D \subset \mathbb{R}^2$ or $(x,y,z)\in\mathbb{R}^3$, $F(x,y,z)=0$

$$A(S) = \iint_D \left(\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1}\right) dA$$

$F(x,y,z) = f(x,y) - z = 0$

$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x}$, $\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y}$, $\frac{\partial F}{\partial z} = -1$

$(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 = 1 = (\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial y})^2 + (\frac{\partial F}{\partial z})^2 = \|\nabla F\|^2$

$$A(S) = \iint_D \|\nabla F\|\, dA$$

Section 7.5

2. $f(x,y,z) = z+6$, $\Phi(u,v) = (u, \frac{v}{3}, v)$, $u \in [0,2]$, $v \in [0,3]$

$T_u = (1, 0, 0)$

$T_v = (0, \frac{1}{3}, 1)$

$T_u \times T_v = (0, -1, \frac{1}{3})$

$\|T_u \times T_v\| = \sqrt{1^2 + \frac{1}{9}} = \frac{\sqrt{10}}{3}$

$f = z+6 = v+6$

$\iint f\,dS = \int_0^3\int_0^2 (v+6)\frac{\sqrt{10}}{3}\,du\,dv$

$= 2 \cdot \frac{\sqrt{10}}{3}\int_0^3 (v+6)\,dv$

$= 15\sqrt{10}$

3. $\iint_S (3x - 2y + z)\,dS$, S: $2x + 3y + z = 6$

$z = 6 - 2x - 3y$

$T_x \times T_y = (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1) = (2, 3, 1)$

$\|T_x \times T_y\| = \sqrt{4+9+1} = \sqrt{14}$

$\iint_D (3x - 2y + 6 - 2x - 3y)\sqrt{14}\,dx\,dy$

$= \sqrt{14}\int_0^3\int_0^{\frac{6-2x}{3}} (3x - 2y + 6 - 2x - 3y)\,dx\,dy$

$= \sqrt{14}\int_0^3 (6+x)(\frac{6-2x}{3}) - \frac{5}{2}(\frac{6-2x}{3})^2\,dx$

$= 11\sqrt{14}$

4. $\iint_S (x+z)\,dS \quad , S: y^2+z^2=4 \quad , x \in [0,5]$

$\phi = (x,\ 2\cos\theta,\ 2\sin\theta)$

let ϕ be parametrization : $x \in [0,5]$
$\theta \in [0,2\pi]$

$T_x = (1,0,0)$

$T_\theta = (0,-2\sin\theta, 2\cos\theta)$

$T_x \times T_\theta = (0,-2\cos\theta,-2\sin\theta)$

$\|T_x \times T_\theta\| = 2$

$\iint_S (x+z)\,dS = \int_0^{2\pi}\int_0^5 (x+2\sin\theta)\,2\cdot dx\,d\theta$

$= \int_0^{2\pi} 2\left(\frac{25}{2}+10\sin\theta\right)d\theta$

$= 25(2\pi)$

$= 50\pi$

6. $\iint_S (x^2z+y^2z)\,dS \quad , S: z=4+x+y \ , \ x^2+y^2=4$

$T_x \times T_y = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = (-1,-1,1)$

$\|T_x \times T_y\| = \sqrt{1+1+1} = \sqrt{3}$

$\iint_S z(x^2+y^2)\sqrt{3}\,dx\,dy$

$x = \alpha\cos\theta \qquad \alpha \in [0,2]$
$y = \alpha\sin\theta \qquad \theta \in [0,2\pi]$

$\int_0^{2\pi}\int_0^2 (4+\alpha\cos\theta+\alpha\sin\theta)\sqrt{3}\,\alpha^2\,\alpha\,d\alpha\,d\theta$

$= 32\sqrt{3}\,\pi$

7. $z=0,\ y=0,\ x+z=1,\ x=y$

$S = S_1 + S_2 + S_3 + S_4$

$S_1 = \langle u, v, 0\rangle \quad , \ 0 \le u \le 1,\ 0 \le v \le u$

$S_2 = \langle u, 0, v\rangle \quad , \ 0 \le u \le 1,\ 0 \le v \le 1-u$

$S_3 = \langle u, u, v\rangle \quad , \ 0 \le u \le 1,\ 0 \le v \le 1-u$

$S_4 = \langle u, v, 1-u\rangle \quad , \ 0 \le u \le 1,\ 0 \le v \le u$

$$I(S_1) = \iint_{S_1} uv\, dS$$

$$\begin{aligned} T_u &= \langle 1,0,0\rangle \\ T_v &= \langle 0,1,0\rangle \end{aligned} \Rightarrow T_u \times T_v = \langle 0,0,1\rangle \Rightarrow \|T_u \times T_v\| = 1$$

$$I(S_1) = \int_0^1 \int_0^u uv\, dv\, du = \int_0^1 \left(\frac{uv^2}{2}\right)\Big|_0^u = \frac{1}{2}$$

$\because y=0 \ \therefore\ I(S_2) = 0$

$$I(S_3) = \iint_{S_3} u^2\, dS = \int_0^1 \int_0^{1-u} u^2 \sqrt{2}\, dv\, du = \frac{\sqrt{2}}{12}$$

$$\begin{aligned} T_u &= \langle 1,1,0\rangle \\ T_v &= \langle 0,0,1\rangle \end{aligned} \Rightarrow T_u \times T_v = \langle 1,-1,0\rangle \Rightarrow \|T_u \times T_v\| = \sqrt{2}$$

$$I(S_4) = \iint_{S_4} uv\, dS = \int_0^1 \int_0^u uv\sqrt{2}\, dv\, du = \frac{\sqrt{2}}{6}$$

$$\begin{aligned} T_u &= \langle 1,0,-1\rangle \\ T_v &= \langle 0,1,0\rangle \end{aligned} \Rightarrow T_u \times T_v = \langle 1,0,1\rangle \Rightarrow \|T_u \times T_v\| = \sqrt{2}$$

$$I(S) = I(S_1) + I(S_2) + I(S_3) + I(S_4) = \frac{2+\sqrt{2}}{4}$$

#18 ?

23. $\Phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$, $S: x = x(u,v),\ y = y(u,v),\ z = z(u,v)$

a).

$$\frac{\partial \Phi}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = T_u$$

$$\frac{\partial \Phi}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = T_v$$

$$E = \left\| \frac{\partial \Phi}{\partial u} \right\|^2, \quad F = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v}, \quad G = \left\| \frac{\partial \Phi}{\partial v} \right\|^2$$

$$\sqrt{EG - F^2} = \sqrt{\left\| \frac{\partial \Phi}{\partial u} \right\|^2 \left\| \frac{\partial \Phi}{\partial v} \right\|^2 - \left(\frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} \right)^2}$$

$$\frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} = \left\| \frac{\partial \Phi}{\partial u} \right\| \left\| \frac{\partial \Phi}{\partial v} \right\| \cos\theta$$

$$\sqrt{EG - F^2} = \sqrt{\left(\left\| \frac{\partial \Phi}{\partial u} \right\| \left\| \frac{\partial \Phi}{\partial v} \right\| \right)^2 (1 - \cos^2\theta)}$$

$$= \sqrt{\left(\left\| \frac{\partial \Phi}{\partial u} \right\| \left\| \frac{\partial \Phi}{\partial v} \right\| \sin\theta \right)^2}$$

$$= \left\| \frac{\partial \Phi}{\partial u} \right\| \left\| \frac{\partial \Phi}{\partial v} \right\| \sin\theta$$

$$= \|T_u\| \|T_v\| \sin\theta$$

$$= \|T_u \times T_v\|$$

b) $\because$ $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial v}$ are orthogonal

$\therefore F = 0$

$$\sqrt{EG - F^2} = \sqrt{EG} = \left\| \frac{\partial \Phi}{\partial u} \right\| \left\| \frac{\partial \Phi}{\partial v} \right\|$$

$$A(S) = \iint_D \left\| \frac{\partial \Phi}{\partial u} \right\| \left\| \frac{\partial \Phi}{\partial v} \right\| \, du\, dv$$

c) $\Phi(\theta, \phi) = (a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi)$

$$\frac{\partial \Phi(u,v)}{\partial \theta} = (-a\sin\theta\sin\phi,\ a\cos\theta\sin\phi,\ 0)$$

$$\frac{\partial \Phi(u,v)}{\partial \phi} = (a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -a\sin\phi)$$

$$E = \left\| \frac{\partial \Phi}{\partial u} \right\|^2 = a^2 \sin^2\phi, \quad F = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} = 0, \quad G = \left\| \frac{\partial \Phi}{\partial v} \right\|^2 = a^2$$

$$\sqrt{EG - F^2} = \sqrt{a^4 \sin^2\phi} = a^2 \sin\phi$$

$$A(S) = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \, d\phi \, d\theta$$

24. $\Phi : D \to \mathbb{R}^3$, $J(\Phi) = \frac{1}{2}\iint_D \left\| \frac{\partial \Phi}{\partial u} \right\|^2 + \left\| \frac{\partial \Phi}{\partial v} \right\|^2)\, du\, dv$

$$A(\Phi) = \iint_D \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| du\, dv$$

$$= \iint_D \left\| \frac{\partial \Phi}{\partial u} \right\| \left\| \frac{\partial \Phi}{\partial v} \right\| \sin\theta \, du\, dv$$

$$(\|T_u\| - \|T_v\|)^2 \geq 0$$

$$\|T_u\|^2 + \|T_v\|^2 \geq 2\|T_u\|\|T_v\|$$

$$\frac{1}{2}(\|T_u\|^2 + \|T_v\|^2) \geq \|T_u\|\|T_v\| \geq \|T_u\|\|T_v\| \sin\theta$$

$$\iint_D \frac{1}{2}(\|T_u\|^2 + \|T_v\|^2)\, du\, dv \geq \iint_D \|T_u\|\|T_v\| \sin\theta \, du\, dv$$

$$J(\Phi) \geq A(\Phi)$$