Tina Chin — Mar. 24, 2017

MATH 255 T|F

2/2 Excellent!

HW#6: sec 7.6 #1, 4, 5, 6, 9, 17, 22

sec 7.7 #1, 6, 7, 9, 10

sec 7.6

1. $z = 1-x^2-y^2 \quad z \geq 0 \quad F(x,y,z) = (2x, 2y, z)$ with unit disc in xy-plane

$u = x \quad y = v \quad z = 1-u^2-v^2 \quad T_u = (1,0,-2u) \quad T_v = (0,1,-2v)$

$T_u \times T_v = (2u, 2v, 1) \quad \iint_S F \cdot dS = \iint_D (2u, 2v, 1-u^2-v^2)\cdot(2u,2v,1)\,du\,dv$

$= \iint_D 4u^2+4v^2+1-u^2-v^2\,du\,dv = \iint_D 3u^2+3v^2+1\,du\,dv$

$u = r\cos\theta \quad v = r\sin\theta \quad z = 1-(u^2+v^2) = 1-r^2 \quad dS = r\,dr\,d\theta$

$\iint_S F\cdot dS = \int_0^{2\pi}\int_0^1 (3r^2\cos^2\theta + 3r^2\sin^2\theta + 1)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r(3r^2+1)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 3r^3+r\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{3}{4}r^4+\frac{r^2}{2}\right]_0^1 d\theta = \int_0^{2\pi}\frac{3}{4}+\frac{1}{2}\,d\theta = \left[\frac{5\pi}{4}\theta\right]_0^{2\pi} = \boxed{\frac{5\pi}{2}}$

4. $F(x,y,z) = (2x, -2y, z^2) \quad S: x^2+y^2=4 \quad z \in [0,1]$

$\Phi(z,\theta) = (2\cos\theta, 2\sin\theta, z) \quad T_z = (0,0,1) \quad T_\theta = (-2\sin\theta, 2\cos\theta, 0)$

$T_z \times T_\theta = (-2\cos\theta, -2\sin\theta, 0) \qquad F(\Phi(z,\theta)) = (4\cos\theta, -4\sin\theta, z^2)$

$\iint_S F\cdot dS = \int_0^{2\pi}\int_0^1 (4\cos\theta, -4\sin\theta, z^2)\cdot(-2\cos\theta, -2\sin\theta, 0)\,d\theta\,dz = \int_0^1\int_0^{2\pi} -8\cos^2\theta + 8\sin^2\theta\,d\theta\,dz$

$= \int_0^1\int_0^{2\pi} -8(\cos^2\theta - \sin^2\theta)\,d\theta\,dz = \int_0^1\int_0^{2\pi} -8\cos 2\theta\,d\theta\,dz = \int_0^1 -8\left[\frac{1}{2}\sin 2\theta\right]_0^{2\pi} dz$

$= -4\int_0^1 \sin 4\pi - \sin 0\,dz = -4\int_0^1 0\,dz = \boxed{0}$

5. $T(x,y,z) = 3x^2+3z^2 \quad x^2+z^2=2, \; 0 \leq y \leq 2, \; k=1$

$F = -\nabla T(x,y,z) = (-6x, 0, -6z) \qquad \Phi(y,\theta) = (\sqrt{2}\cos\theta, y, \sqrt{2}\sin\theta)$

$T_y = (0,1,0) \quad T_\theta = (-\sqrt{2}\sin\theta, 0, \sqrt{2}\cos\theta) \quad T_\theta \times T_y = (-\sqrt{2}\cos\theta, 0, -\sqrt{2}\sin\theta)$

$F(\Phi(y,\theta)) = (-6\sqrt{2}\cos\theta, 0, -6\sqrt{2}\sin\theta) \quad \iint_S F\cdot dS = \int_0^2\int_0^{2\pi}(-6\sqrt{2}\cos\theta, 0, -6\sqrt{2}\sin\theta)\cdot$

$(-\sqrt{2}\cos\theta, 0, -\sqrt{2}\sin\theta)\,d\theta\,dy = \int_0^2\int_0^{2\pi} 12\cos^2\theta + 0 + 12\sin^2\theta\,d\theta\,dy = \int_0^2\int_0^{2\pi} 12\,d\theta\,dy$

$= \int_0^2 [12\theta]_0^{2\pi}\,dy = \int_0^2 24\pi\,dy = [24\pi y]_0^2 = \boxed{48\pi}$

6. $T(x,y,z) = x$

$F = -\nabla T(x,y,z) = (-1,0,0)$ unit sphere: $x^2+y^2+z^2=1 \quad n = (x,y,z)$

$\iint_S F\cdot dS = \iint_S F\cdot n\,dS = \iint_S (-1,0,0)\cdot(x,y,z)\,dS = \iint_S -x\,dS = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -x\,dy\,dx$

$= \int_{-1}^1 -\left[\frac{x^2}{2}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = -\int_{-1}^1 \frac{1-x^2}{2} - \frac{1-x^2}{2}\,dx = -\int_{-1}^1 0\,dx = \boxed{0}$ The flux across the unit sphere is 0.

9. $S: x^2+y^2+3z^2=1, \; z \leq 0 \quad F = (y, -x, 2x^3y^2)$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 2x^3y^2 \end{vmatrix} = (2yzx^3, -3x^2zy^2, -2)$$

$z = \sqrt{\frac{1-x^2-y^2}{3}} = \frac{1}{\sqrt{3}}\sqrt{1-x^2-y^2}$ $\quad T_x = \left(1, 0, \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)(1-x^2-y^2)^{-1/2}(-2x)\right) = \left(1, 0, \frac{-x}{\sqrt{3}\sqrt{1-x^2-y^2}}\right)$

$T_y = \left(0, 1, \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)(1-x^2-y^2)^{-1/2}(-2y)\right) = \left(0, 1, \frac{-y}{\sqrt{3}\sqrt{1-x^2-y^2}}\right)$

$T_x \times T_y = \left(\frac{x}{\sqrt{3}\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{3}\sqrt{1-x^2-y^2}}, 1\right)$

$\iint_S (\nabla \times F)\, dS = \iint_S (2yzx^3, -3x^2zy^2, -2) \cdot \left(\frac{x}{\sqrt{3}\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{3}\sqrt{1-x^2-y^2}}, 1\right) dy\,dx$

$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2yzx^4}{\sqrt{3}\sqrt{1-x^2-y^2}} - \frac{3x^2zy^3}{\sqrt{3}\sqrt{1-x^2-y^2}} - 2\, dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2yx^4}{3} - x^2y^3 - 2\, dy\,dx$ ← substitute z

$= \int_{-1}^{1}\left[\frac{1}{3}y^2x^4 - \frac{1}{4}x^2y^4 - 2y\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^{1}\left(\frac{1}{3}x^4(1-x^2) - \frac{1}{4}x^2(1-x^2)^2 - 2\sqrt{1-x^2} - \frac{1}{3}x^4(1-x^2)\right.$

$\left. + \frac{1}{4}x^2(1-x^2)^2 - 2\sqrt{1-x^2}\right) dx = \int_{-1}^{1} -4\sqrt{1-x^2}\, dx = -4\int_{-1}^{1}\sqrt{1-x^2}\, dx$

$= -4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_{-1}^{1} = -4\left(\frac{1}{2}\sqrt{0} + \frac{1}{2}\sin^{-1}(1) - 0 - \frac{1}{2}\sin^{-1}(-1)\right)$

$= -4\left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right)\right) = \boxed{-2\pi}$

17. surface integral: $\iint_S F \cdot dS = \iint_D F \cdot (T_x \times T_y)\, dx\,dy$

parametrization of a cylinder: $\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$

$T_\theta = (-\sin\theta, \cos\theta, 0)$ $\quad T_z = (0,0,1)$ $\quad T_\theta \times T_z = (\cos\theta, +\sin\theta, 0)$

$\iint_D F \cdot (T_x \times T_y)\, dx\,dy = \int_0^{2\pi}\int_{z=a}^{z=b} F \cdot (\cos\theta, \sin\theta, 0)\, dz\,d\theta$

22.a. $F(x,y,z) = (x,y)$

$z = \sqrt{1-x^2-y^2}$ $\quad T_x = \left(1, 0, \frac{1}{2}(1-x^2-y^2)^{-1/2}(-2x)\right) = \left(1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}\right)$

$T_y = \left(0, 1, \frac{1}{2}(1-x^2-y^2)^{-1/2}(-2y)\right) = \left(0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}\right)$

$T_x \times T_y = \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right)$

$\iint_S F \cdot dS = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x,y) \cdot \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right) dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{x^2}{\sqrt{1-x^2-y^2}} + \frac{y^2}{\sqrt{1-x^2-y^2}}\, dy\,dx$

$x = r\cos\theta \quad y = r\sin\theta \quad dy\,dx = r\,dr\,d\theta$

$= \int_0^1\int_0^{2\pi} \frac{r^3}{\sqrt{1-r^2}}\, d\theta\, dr = \int_0^1 \frac{2\pi r^3}{\sqrt{1-r^2}}\, dr = 2\pi\int_0^1 \frac{r^3}{\sqrt{1-r^2}}\, dr \quad u = r^2 \quad \frac{du}{2r} = dr$

$= \pi\int_0^1 \frac{u}{\sqrt{1-u}}\, du = \pi\left[\int_0^1 \frac{1}{\sqrt{1-u}} - \frac{1-u}{\sqrt{1-u}}\, du\right] = \pi\int_0^1 \frac{1}{\sqrt{1-u}} - \sqrt{1-u}\, du$

$= \pi\left[-2(1-r^2)^{1/2} + \frac{2}{3}(1-r^2)^{3/2}\right]_0^1 = \pi\left[(0-0) - \left(-2+\frac{2}{3}\right)\right] = \pi\left(2 - \frac{2}{3}\right) = \boxed{\frac{4}{3}\pi}$

22.b. $F(x,y,z) = (y,x)$

$\iint_S F \cdot ds = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y,x) \cdot \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right) dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-x^2-y^2}} + \frac{yx}{\sqrt{1-x^2-y^2}}\, dy\,dx$

$= 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-x^2-y^2}}\, dy\,dx$

$= 2\int_0^1\int_0^{2\pi} \frac{r^2\cos\theta\sin\theta}{\sqrt{1-r^2}}\, d\theta\, dr \quad u = \sin\theta \quad \frac{du}{\cos\theta} = d\theta \quad = 2\int_0^1\int_0^{2\pi} \frac{r^2 u}{\sqrt{1-r^2}}\, du\,dr$

$= 2\int_0^1 \frac{1}{2}\left[\frac{r^2\sin^2\theta}{\sqrt{1-r^2}}\right]_0^{2\pi} dr = \int_0^1 0\, dr = \boxed{0}$

22.c. $F_1 = (x,y)$ $\quad F_2 = (y,x)$

$\nabla \times F_1 = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = (0,0,0)$ $\qquad \nabla \times F_2 = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = (0,0,0)$

$\iint_S (\nabla \cdot F)\, dS$ for both F_1 & F_2 = $\boxed{0}$

$\int_C F\cdot ds = \int_C F\cdot n\,ds$ the normal to surface C – the unit circle in xy plane – is

$n = (0,0,1)$

$F_1 \cdot n = (x,y)\cdot(0,0,1) = 0 \qquad F_2\cdot n = (y,x)\cdot(0,0,1) = 0$

$\int_C F\cdot ds = 0$ for both F_1 & F_2

sec 7.7 1. $\Phi(u,v) = (u\cos v, u\sin v, bv)$ where $b \neq 0$

$H(p) = \dfrac{G\ell + En - 2Fm}{2(EG-F^2)} \qquad T_u = (\cos v, \sin v, 0) \qquad T_u \times T_v = (b\sin v, -b\cos v, u\cos^2 v + u\sin^2 v)$

$T_v = (-u\sin v, u\cos v, b)$

$$N = \frac{(b\sin v, -b\cos v, u)}{\sqrt{b^2\sin^2 v + b^2\cos^2 v + u^2}} = \frac{(b\sin v, -b\cos v, u)}{\sqrt{b^2+u^2}}$$

$E = 1 \qquad F = -u\sin v\cos v + u\cos v\sin v = 0 \qquad G = u^2 + b^2$

$T_{uu} = (0,0,0) \qquad \ell = N\cdot T_{uu} = 0$

$$T_{uv} = (-\sin v, \cos v, 0) \qquad m = N\cdot T_{uv} = \frac{(b\sin v, -b\cos v, u)}{\sqrt{b^2+u^2}}\cdot(-\sin v, \cos v, 0) = \frac{-b}{\sqrt{b^2+u^2}}$$

$$T_{vv} = (-u\cos v, -u\sin v, 0) \qquad n = N\cdot T_{vv} = \frac{(b\sin v, -b\cos v, u)}{\sqrt{b^2+u^2}}\cdot(-u\cos v, -u\sin v, 0) = 0$$

$$H = \frac{(u^2+b^2)(0) + 1(0) - 2(0)\left(\frac{-b}{\sqrt{b^2+u^2}}\right)}{2(EG-F^2)} = \boxed{0} ✓$$

$$K = \frac{\ell n - m^2}{EG-F^2} = \frac{0(0) - \left(\frac{-b}{\sqrt{b^2+u^2}}\right)^2}{u^2+b^2-0^2} = \frac{\frac{-b^2}{u^2+b^2}}{u^2+b^2} = \boxed{\frac{-b^2}{(u^2+b^2)^2}} ✓$$

b. $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$

$\Phi(u,v) = (a\cos u\sin v, a\sin u\sin v, c\cos v)$

$T_u = (-a\sin u\sin v, a\cos u\sin v, 0) \qquad T_u \times T_v = (-ac\cos u\sin^2 v, -ac\sin u\sin^2 v,$

$T_v = (a\cos u\cos v, a\sin u\cos v, -c\sin v) \qquad -a^2\sin^2 u\sin v\cos v - a^2\cos^2 u\cos v\sin v)$

$= (-ac\cos u\sin^2 v, -ac\sin u\sin^2 v, -a^2\sin v\cos v)$

$$N = \frac{(-ac\cos u\sin^2 v, -ac\sin u\sin^2 v, -a^2\sin v\cos v)}{\sqrt{a^2c^2\cos^2 u\sin^4 v + a^2c^2\sin^2 u\sin^4 v + a^4\sin^2 v\cos^2 v}} = \frac{(-ac\cos u\sin^2 v, -ac\sin u\sin^2 v, -a^2\sin v\cos v)}{\sqrt{a^2c^2\sin^4 v + a^4\sin^2 v\cos^2 v}}$$

$E = a^2\sin^2 v \qquad F = 0 \qquad G = a^2\cos^2 v + c^2\sin^2 v$

$$T_{uu} = (-a\cos u\sin v, -a\sin u\sin v, 0) \qquad \ell = \frac{a^2\sin^3 v\, c}{\sqrt{a^2c^2\sin^4 v + a^4\sin^2 v\cos^2 v}}$$

$T_{uv} = (-a\sin u\cos v, a\cos u\cos v, 0) \qquad m = 0$

$$T_{vv} = (-a\cos u\sin v, -a\sin u\sin v, -c\cos v) \qquad n = \frac{ca^2\sin^3 v + a^2c\sin v\cos^2 v}{\sqrt{a^2c^2\sin^4 v + a^4\sin^2 v\cos^2 v}} = \frac{a^2c\sin v}{\sqrt{a^2c^2\sin^4 v + a^4\sin^2 v\cos^2 v}}$$

$$K = \frac{\dfrac{a^2 c \sin^3 v}{a\sin v\sqrt{c^2\sin^2 v + a^2\cos^2 v}}\left(\dfrac{a^2 c\sin v}{a\sin v\sqrt{c^2\sin^2 v + a^2\cos^2 v}}\right)}{a^2\sin^2 v\,(a^2\cos^2 v + c^2\sin^2 v)} = \boxed{\frac{a^4c^2\sin^4 v}{[a^2\sin^2 v\,(a^2\cos^2 v + c^2\sin^2 v)]^2}}$$

7. $\frac{1}{2\pi}\iint_S K\,dA = 2$

$T_u \times T_v = (-ac\cos u\sin^2 v,\ -ac\sin u\sin^2 v,\ -a^2\sin v\cos v)$

$\|T_u \times T_v\| = \sqrt{a^2c^2\cos^2 u\sin^4 v + a^2c^2\sin^2 u\sin^4 v + a^4\sin^2 v\cos^2 v} = \sqrt{a^2c^2\sin^4 v + a^4\sin^2 v\cos^2 v}$

$= a\sin v\sqrt{c^2\sin^2 v + a^2\cos^2 v}$

$$K = \frac{a^4c^2\sin^4 v}{a^4\sin^4 v\,(a^2\cos^2 v + c^2\sin^2 v)^2} = \frac{a^4c^2}{a^4(a^2\cos^2 v + c^2\sin^2 v)^2}$$

$$\frac{1}{2\pi}\iint_S K\,dA = \frac{1}{2\pi}\iint_S \left(\frac{a^4c^2}{a^4(a^2\cos^2 v + c^2\sin^2 v)^2}\right) a\sin v\sqrt{c^2\sin^2 v + a^2\cos^2 v}\ dv$$

$$= \frac{1}{2\pi}\int_0^{\pi}\int_0^{2\pi}\frac{a^5c^2\sin v}{a^4(a^2\cos^2 v + c^2\sin^2 v)^{3/2}}\,du\,dv = \int_0^{\pi}\frac{ac^2\sin v}{(a^2\cos^2 v + c^2\sin^2 v)^{3/2}}\,dv$$

$$= \int_0^{\pi}\frac{ac^2\sin v}{(a^2\cos^2 v + c^2(1-\cos^2 v))^{3/2}}\,dv = ac^2\int_0^{\pi}\frac{\sin v}{(a^2\cos^2 v + c^2 - c^2\cos^2 v)^{3/2}}\,dv$$

$$= \frac{ac^2}{(a^2-c^2)^{3/2}}\int_0^{\pi}\frac{\sin v}{\left(\cos^2 v + \frac{c^2}{(a^2-c^2)}\right)^{3/2}}\,dv \qquad w_1 = \cos v \quad \frac{dw_1}{-\sin v} = dv \quad = \frac{ac^2}{(a^2-c^2)^{3/2}}\int_{-1}^{1}\frac{1}{\left(w^2 + \left(\sqrt{\frac{c^2}{a^2-c^2}}\right)^2\right)^{3/2}}\,dw_1$$

$w_2 = \sqrt{\frac{c^2}{a^2-c^2}}\tan\theta$

$$= \frac{ac^2}{(a^2-c^2)^{3/2}}\int_{-1}^{1}\frac{\sec^2\theta}{\left(\frac{c^2}{a^2-c^2}\right)(\sec^2\theta)^{3/2}}\,d\theta = \frac{ac^2}{(a^2-c^2)^{3/2}}\left[\sqrt{\frac{w^2(a^2-c^2)}{c^2 + w^2(a^2-c^2)}}\right]_{-1}^{1} = \boxed{2}$$

9. $\Phi(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + u^2 v,\ u^2 - v^2\right)$

$T_u = (1 - u^2 + v^2,\ 2uv,\ 2u) \qquad T_v = (2uv,\ 1 - v^2 + u^2,\ -2v)$

$E = (1-u^2+v^2)(1-u^2+v^2) + 4u^2v^2 + 4u^2 = 1 - u^2 + v^2 - u^2 + u^4 - u^2v^2 + v^2 - u^2v^2 + v^4 + 4u^2v^2 + 4u^2$

$= 2u^2 + 1 + 2v^2 + v^4 + u^4 + 2u^2v^2 = (u^2+v^2+1)^2$

$F = (1-u^2+v^2)2uv + 2uv(1-v^2+u^2) - 4uv = 0$

$G = 4u^2v^2 + (1-v^2+u^2)(1-v^2+u^2) + 4v^2 = 1 - v^2 + u^2 - v^2 + v^4 - u^2v^2 + u^2 - u^2v^2 + u^4 + 4u^2v^2 + 4v^2$

$= 2v^2 + 1 + 2u^2 + u^4 + v^4 + 2u^2v^2 = (u^2+v^2+1)^2$

$\|T_u \times T_v\|^2 = EG - F^2 = (u^2+v^2+1)^4$

$1-\cancel{v^2}+\cancel{u^2}-\cancel{u^2}+u^2v^2-u^4+\cancel{v^2}-v^4+u^2v^2$

$T_u \times T_v = (-4uv^2-2u+2uv^2-2u^3,\ 4u^2v+2v-2vu^2+2v^3,\ (1-u^2+v^2)(1-v^2+u^2)-4u^2v^2)$

$= (-2uv^2-2u-2u^3,\ 2u^2v+2v+2v^3,\ -2u^2v^2-u^4-v^4+1)$

$N = \dfrac{(-2uv^2-2u-2u^3,\ 2u^2v+2v+2v^3,\ -2u^2v^2-u^4-v^4+1)}{(u^2+v^2+1)^2}$

$l = \dfrac{(-2uv^2-2u-2u^3,\ 2u^2v+2v+2v^3,\ -2u^2v^2-u^4-v^4+1)}{(u^2+v^2+1)^2} \cdot (-2u, 2v, 2)$

$= \dfrac{1}{(u^2+v^2+1)^2}\left[4u^2v^2+4u^2+4u^4+\cancel{4u^2v^2}+4v^2+4v^4-\cancel{4u^2v^2}-2u^4-2v^4+2\right]$

$= \dfrac{1}{(u^2+v^2+1)^2}\left[2v^4+2u^4+4u^2+4v^2+4u^2v^2+2\right] = \dfrac{2}{(u^2+v^2+1)^2}(v^4+u^4+2u^2+2v^2+2v^2v^2+1)$

$= \dfrac{2}{(u^2+v^2+1)^2}(u^2+v^2+1)^2 = 2$

$m = \dfrac{(-2uv^2-2u-2u^3,\ 2u^2v+2v+2v^3,\ -2u^2v^2-u^4-v^4+1)}{(u^2+v^2+1)^2} \cdot (2v, 2u, 0)$

$= \dfrac{1}{(u^2+v^2+1)^2}\left[-\cancel{4uv^3}-4uv-\cancel{4u^3v}+\cancel{4u^3v}+4uv+\cancel{4uv^3}\right] = 0$

$n = \dfrac{(-2uv^2-2u-2u^3,\ 2u^2v+2v+2v^3,\ -2u^2v^2-u^4-v^4+1)}{(u^2+v^2+1)^2} \cdot (2u, -2v, -2)$

$= \dfrac{1}{(u^2+v^2+1)^2}\left[-4u^2v^2-4u^2-4u^4-\cancel{4u^2v^2}-4v^2-4v^4+\cancel{4u^2v^2}+2u^4+2v^4-2\right]$

$= \dfrac{-2}{(u^2+v^2+1)^2}(u^4+v^4+2u^2+2v^2+2u^2v^2+1) = -2$

$H(p) = \dfrac{Gl+En-2Fm}{2(EG-F^2)} = \dfrac{(u^2+v^2+1)^2(2)+(u^2+v^2+1)^2(-2)+2(0)(0)}{2((u^2+v^2+1)^4-0)} = \boxed{0 \checkmark}$

10. $\Phi(\theta,\phi) = ((R+\cos\phi)\cos\theta,\ (R+\cos\phi)\sin\theta,\ \sin\phi)$ $\quad 0 \le \phi \le 2\pi \quad 0 \le \theta \le 2\pi$

$T_\theta = (-R\sin\theta-\cos\phi\sin\theta,\ R\cos\theta+\cos\phi\cos\theta,\ 0) \qquad T_\phi = (-\sin\phi\cos\theta,\ -\sin\phi\sin\theta,\ \cos\phi)$

$E = (-R\sin\theta-\cos\phi\sin\theta)^2+(R\cos\theta+\cos\phi\cos\theta)^2 = R^2\sin^2\theta+2R\sin^2\theta\cos\phi+\cos^2\phi\sin^2\theta$

$+R^2\cos\theta+2R\cos^2\theta\cos\phi+\cos^2\phi\cos^2\theta = R^2+2R\cos\phi+\cos^2\phi = (\cos\phi+R)^2$

$F = (-R\sin\theta-\cos\phi\sin\theta)(-\sin\phi\cos\theta)+(R\cos\theta+\cos\phi\cos\theta)(-\sin\phi\sin\theta) = R\sin\theta\sin\phi\cos\theta$

$+\cos\phi\cos\theta\sin\phi\sin\theta-R\cos\theta\sin\theta\sin\phi-\cos\phi\cos\theta\sin\phi\sin\theta = 0$

$G = \sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta+\cos^2\phi = \sin^2\phi+\cos^2\phi = 1$

$\|T_\theta \times T_\phi\|^2 = EG-F^2 = (\cos\phi+R)^2(1)-0 = (\cos\phi+R)^2$

$T_\theta \times T_\phi = (R\cos\theta\cos\phi+\cos^2\phi\cos\theta,\ R\sin\theta\cos\phi+\cos^2\phi\sin\theta,\ R\sin^2\theta\sin\phi+\cos\phi\sin^2\theta\sin\phi$

$+R\cos^2\theta\sin\phi+\cos^2\theta\cos\phi\sin\phi) = (\cos\theta\cos\phi(R+\cos\phi),\ \cos\phi\sin\theta(R+\cos\phi),\ (R+\cos\phi)\sin\phi)$

$l = \dfrac{(\cos\theta\cos\phi(R+\cos\phi),\ \cos\phi\sin\theta(R+\cos\phi),\ (R+\cos\phi)\sin\phi)}{\cos\phi+R} \cdot (-R\cos\theta-\cos\phi\cos\theta,\ -R\sin\theta-\cos\phi\sin\theta,\ 0)$

$= \dfrac{1}{\cos\phi+R}\left[+(R+\cos\phi)^2\cos^2\theta\cos\phi+(R+\cos\phi)^2\cos\phi\sin^2\theta\right] = \dfrac{1}{\cos\phi+R}(+(R+\cos\phi)^2\cos\phi) = \dfrac{+\cos\phi}{(R+\cos\phi)}$

$m = \vec{N}\cdot(\sin\phi\sin\theta,\ -\sin\phi\cos\theta,\ 0) = \dfrac{1}{\cos\phi+R}\left[\cos\theta\cos\phi\sin\phi\sin\theta(R+\cos\phi)-\sin\phi\cos\phi\sin\theta\cos\theta\cdot\right.$

$\left.(R+\cos\phi)\right] = \dfrac{1}{\cos\phi+R}(0) = 0$

$n = \vec{N}\cdot(-\cos\phi\cos\theta,\ -\cos\phi\sin\theta,\ -\sin\phi) = \dfrac{1}{\cos\phi+R}\left[+\cos^2\phi\cos^2\theta(R+\cos\phi)+\cos^2\phi\sin^2\theta(R+\cos\phi)\right.$

$$+(R+\cos\phi)\sin^2\phi\,] = \frac{1}{\cos\phi+R}\left[+\cos^2\phi(R+\cos\phi)+(R+\cos\phi)\sin^2\phi\right] = \frac{(R+\cos\phi)}{R+\cos\phi} = 1$$

$$K(p) = \frac{\ell n-m^2}{EG-F^2} = \frac{\cos\phi(R+\cos\phi)(1)}{(\cos\phi+R)^2} = \boxed{\frac{\cos\phi}{\cos\phi+R}}$$

Gauss-Bonnet Theorem: $\frac{1}{2\pi}\iint_S K\,dA = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} K\cdot \| T_\theta \times T_\phi \| \, d\theta\, d\phi$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\cos\phi}{\cos\phi+R}\right)(\cos\phi+R)\,d\theta\,d\phi = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\cos\phi\,d\theta\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\cos\phi\,\theta\right]_0^{2\pi}d\phi = \int_0^{2\pi}\cos\phi\,d\phi = \left[\sin\phi\right]_0^{2\pi} = 0$$