Adam Murtha
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2/2 Excellent!

MATH 255
PROF. Park

Homework 6

S_1: $1-x^2-y^2$, $z \geq 0$ $\quad S_2: x^2+y^2=1$

$\vec{F}(x,y,z) = (2x, 2y, z)$

$\vec{n}_1 = (2x, 2y, 1)$

$$\iint_{S_1} \vec{F} \cdot d\vec{S}_1 = \iint_D \vec{F} \cdot \vec{n}\, dS = \iint_D 4x^2+4y^2+(1-(x^2+y^2))\,dx\,dy$$

$$\int_0^{2\pi}\int_0^1 (4r^2\cos^2\theta + 4r^2\sin^2\theta + (1-(r^2\cos^2\theta + r^2\sin^2\theta)))\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (4r^2+1-r^2)\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 3r^3+r\,dr\,d\theta =$$

$$\int_0^{2\pi} \frac{3}{4}+\frac{1}{2} = \frac{5}{4}(2\pi) = \frac{10\pi}{4} = \frac{5\pi}{2}$$

$$\iint_{S_2} \vec{F} \cdot d\vec{S}_2 = \iint_{S_2} (0,0,1)\,\vec{F}\cdot\vec{n}\, dS_2 = 0$$

$$\boxed{\text{Surface Integral: } \frac{5\pi}{2}}$$

(4.) $\vec{F}(x,y,z) = (2x, -2y, z^2)$

S_1: $z=1$ $\quad S_2$: $z=0$ $\quad S_3$: $0 < z < 1$

n_1: $(0,0,1)$ $\quad n_2$: $(0,0,0)$ $\quad n_3$: $\left(\frac{x}{2}, \frac{y}{2}, 0\right)$

$F \cdot n_1 = z^2 = 1$

$F \cdot n_2 = z^2 = 0$

$F \cdot n_3 = x^2 - y^2$

$$\iint_{D_1} dx\,dy + \iint_{D_2} x^2-y^2\,dx\,dy =$$

$$4\pi + \int_0^{2\pi}\int_0^2 r(r^2\cos 2\theta)\,dr\,d\theta =$$

$\boxed{4\pi}$

(5) $T(x,y,z) = 3x^2+3z^2$

$x^2+z^2=2$, $0 \le y \le 2$, $K=1$

$$F = -\nabla T = (-6x, 0, -6z)$$

$$n = (x/\sqrt{2}, 0, z/\sqrt{2})$$

$$F \cdot n = \frac{-6}{\sqrt{2}}(x^2+z^2) = -6\sqrt{2}$$

$$\iint_S \vec{F} \cdot dS = -6\sqrt{2} \iint_S dS = -6\sqrt{2}(4\pi\sqrt{2}) = -48\pi$$

directed toward inside or outward

(6) $T(x,y,z) = x$

$x^2+y^2+z^2=5$

$$F = -\nabla T = (-1, 0, 0)$$

$$n = (x, y, z)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n} \sin\phi \, d\phi \, d\theta = \quad F \cdot n = -x$$

$$\int_0^{\pi} \int_0^{2\pi} -\cos\theta \sin\phi \, d\theta \, d\phi = \int_0^{\pi} -\sin\theta \sin^2\phi \Big|_0^{2\pi} d\phi$$

$= \boxed{0}$ → The net heat transfer in & out of the sphere is 0; heat is leaving the sphere at the same rate it enters.

(9) $S: x^2+y^2+3z^2=1$, $z \le 0$; $\vec{F} = (y, -x, zx^3y^2)$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & zx^3y \end{vmatrix} = (zx^3, -2zx^2y, -2)$$

$$\vec{G} = \nabla \times \vec{F} = (zx^3, -2zx^2, -2)$$

$$\vec{n} = \frac{x + y + \sqrt{3}z}{\sqrt{3}}$$

$$\vec{G} \cdot \vec{n} = \frac{1}{\sqrt{3}}(zx^4 - 2zyx^2 - 2\sqrt{3}z)$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D \vec{G} \cdot \vec{n}\, dS = \frac{1}{\sqrt{3}} \iint_D zx^4 - 2zyx^2 - 2\sqrt{3}z\, dD =$$

$$= \frac{1}{\sqrt{3}} \int_0^{\pi/2} \int_0^{2\pi} (\cos\phi \cos^4\theta \sin^4\phi - 2\cos\phi \sin\theta \sin\phi \cos^2\theta \sin\phi - 2\sqrt{3}\cos\phi) \sin\phi\, d\theta\, d\phi$$

$$= \frac{1}{\sqrt{3}} \int_0^{\pi/2} \int_0^{2\pi} \cos\phi \cos^4\theta \sin^5\phi - 2\cos\phi \sin\theta \sin^3\phi \cos^2\theta - 2\sqrt{3}\cos\phi \sin\phi\, d\theta\, d\phi$$

Let $u = \sin\phi$

$du = \cos\phi$

$$\int_0^{2\pi} \int_0^{1} \cos^4\theta\, u^5 - 2\sin\theta \cos^2\theta\, u^3 - 2\sqrt{3}\, u\, du\, d\theta =$$

$$= \frac{1}{\sqrt{3}} \int_0^{2\pi} \cos^4\theta \frac{u^6}{6} - 2\sin\theta \cos^2\theta \frac{u^4}{4} - \sqrt{3} \Big|_0^1 d\theta =$$

$$= \frac{1}{\sqrt{3}} \int_0^{2\pi} \frac{\cos^4\theta}{6} - \frac{\sin\theta\cos^2\theta}{2} - \sqrt{3} = \int_0^{2\pi} \frac{\cos^4\theta}{6\sqrt{3}} - \frac{\sin\theta\cos^2\theta}{2\sqrt{3}} - 1 =$$

$\boxed{2\pi}$

(17) $\iint \vec{F} \cdot d\vec{S} = \iint (F_r)$

$$\boxed{\int_0^1 \int_0^{2\pi} F_r\, r\, d\theta\, dr}$$

(22) (a) $\vec{F}(x,y,z) = (x, y, 0)$

$\vec{n} = (x, y, z)$ $\quad \vec{F} \cdot \vec{n} = x^2 + y^2$

$$\iint_D x^2 + y^2\, dD = \int_0^{\pi/2} \int_0^{2\pi} (\cos^2\theta \sin^2\phi + \sin^2\theta \sin^2\phi) \sin\phi\, d\theta\, d\phi$$

$$= 2\pi \int_0^{\pi/2} 2\sin^3\phi = 4\pi \int_0^{\pi/2} (1 - \cos^2\phi) \sin\phi\, d\phi$$

$u = \cos\phi$

$du = -\sin\phi$

$$2\pi \int_0^1 1 - u^2\, du = 2\pi \left(u - \frac{u^3}{3}\right)\Big|_0^1 = \boxed{\frac{4\pi}{3}}$$

(b) $\vec{F} = (y, x, 0)$ $\quad \vec{F} \cdot \vec{n} = 2xy$

$$\iint_D 2xy = \int_0^{\pi/2} \int_0^{2\pi} 2\cos\theta \sin\theta \sin^3\phi\, d\theta\, d\phi =$$

Let $u = \cos\theta$

$du = \sin\theta$

$$2\int_0^{\pi/2} 0 = \boxed{0}$$

(c) $\nabla \times \vec{F} = 0$ for both (a.) and (b.),

$\vec{C} = (\cos t, \sin t, 0)$

(a) $\int_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0)\, dt = \boxed{0}$

(b) $\int_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt = \boxed{0}$

(7.70) (1) $\Phi(u,v) = (u\cos v, u\sin v, bv),\ b \neq 0$

$\vec{T}_u = (\cos v, \sin v, 0)$ $\quad \vec{T}_v = (-u\sin v, u\cos v, b)$

$\vec{T}_{uu} = (0,0,0)$ $\quad T_{vv} = (-u\cos v, -u\sin v, 1)$

$\vec{T}_{uv} = (-\sin v, \cos v, 0)$

$K(\phi) = \dfrac{\ell n - m^2}{EG - F^2} = \dfrac{0 - b^2}{EG - F^2} = \dfrac{-b^2}{(b^2 + u^2)^2}$

$H = \dfrac{G\ell + En - 2Fm}{2\sqrt{EG - F^2}} = \dfrac{0 + -u\cos v - u\sin v + 1 \quad -\sin v + \cos v =}{2\sqrt{u^2 + v^2}\ \phi}$

$\boxed{0}$

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} + \dfrac{z^2}{c^2} = 1$

$\Phi(u,v) = (a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ c\cos\phi)$

$T_\theta = (-a\sin\theta\sin\phi,\ a\cos\theta\sin\phi,\ 0)$

$\vec{T}_\phi = (a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -c\sin\phi)$

$\vec{T}_\theta \times \vec{T}_\phi = (-ac\sin^2\phi\cos\theta,\ -ac\sin^2\phi\sin\theta,\ -a\sin\phi\cos\phi)$

$\vec{T}_{\theta\theta} = (-a\cos\theta\sin\phi,\ -a\sin\theta\sin\phi,\ 0)$

$\vec{T}_{\theta\phi} = (-a\sin\theta\cos\phi,\ a\cos\theta\cos\phi,\ 0$

$\vec{T}_{\phi\phi} = (-a\cos\theta\sin\phi,\ -a\sin\theta\sin\phi,\ -c\cos\phi)$

$\ell = \dfrac{a^2 c\sin^3\phi}{\sqrt{w}}$ $\qquad w = a^2c^2\sin^4\phi + a^4\sin^2\phi\cos^2\phi$

$m = 0$

$n = \dfrac{a^2 c\sin\phi}{\sqrt{w}}$ $\qquad K = \dfrac{\ell n - m^2}{w} = \dfrac{(a^2c\sin^3\phi)(a^2c\sin^3\phi + a^2 c\sin\phi\cos^2\phi)}{w^2}$

$= \left(\dfrac{c}{c^2\sin^2\phi + a^2\cos^2\phi}\right)^2$

(7.) $\frac{1}{2a}\iint_S K\,dA = \frac{ac^2}{2\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\sin\phi\,d\theta\,d\phi}{(c^2\sin^2\phi+a^2\cos^2\phi)^{3/2}}$

$= ac^2\int_0^{\pi}\frac{\sin\phi}{(c^2\sin^2\phi+a^2\cos^2\phi)^{3/2}}\,d\phi$

$u=\cos\phi$

$du=\sin\phi$

$-ac^2\int_1^0\frac{1}{1(c^2+(a^2-c^2)\cos^2\phi))^{3/2}} = \frac{-a}{\sqrt{a^2-c^2}}\int_1^{2}\frac{dv}{1+\frac{\sqrt{a^2-c^2}\cos\phi}{c}} =$

$=\frac{a}{c}\left(\frac{-2}{\sqrt{a^2+c^2}}\right)=\boxed{2}$

(9.) $\Phi(u,v)=(u-\frac{u^3}{3}+uv^2,\ v-\frac{v^3}{3}+u^2v,\ u^2-v^2)$

$\cancel{\vec{T}_u = (1-v^2+u^2, 2uv, 2u)}$

$\vec{T}_u = (1-u^2+v^2, 2uv, 2u)$ $\quad \vec{T}_{uu}=(-2u, 2v, 2)$ $\quad \vec{T}_{uv}=(2v, 2u, 0)$

$\vec{T}_v = (2uv, 1-v^2+u^2, -2v)$ $\quad \vec{T}_{vv}=(2u, -2v, -2)$

$\vec{T}_u\times\vec{T}_v=(1+u^2+v^2)(-2u, 2v, 1-u^2-v^2)$

$E=(1-u^2+v^2)^2+(2uv)^2+(2u)^2 \quad F=0 \quad G=(1-v^2+u^2)^2+(2v)^2+(2uv)^2$

$\ell=\frac{2(1+u^2+v^2)^2}{\sqrt{W}} \quad n=\frac{-2(1+u^2+v^2)^2}{\sqrt{W}}$

$H=\frac{G\ell+En-2Fm}{W}$

$H=\frac{\ell}{W}(G-E)$

$=\frac{\ell}{W}(1-v^2+u^2)^2-(1-u^2+v^2)^2+(2v)^2-(2u)^2=$

$4\frac{\ell}{W}(u^2-v^2+v^2-u^2)=0$

$\boxed{H=0}$

10.) $\Phi(u,v) = ((R+\cos\phi)\cos\theta, (R+\cos\phi)\sin\theta, \sin\phi)$

$\vec{T}_\theta = (-(R+\cos\phi)\sin\theta, (R+\cos\phi)\cos\theta, 0)$

$T_\phi = (-\sin\phi\cos\theta, -\sin\phi\sin\theta, \cos\phi)$

$T_\theta \times T_\phi = ((R+\cos\phi)\cos\theta\cos\phi, (R+\cos\phi)\sin\theta\cos\phi, (R+\cos\phi)\sin\phi)$

$W = (R+\cos\phi)^2 = \|T_\theta \times T_\phi\|^2$

$N = \frac{T_\theta \times T_\phi}{W} = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$

$\vec{T}_{\theta\theta} = (-(R+\cos\phi)\cos\theta, -(R+\cos\phi)\sin\theta, 0)$

$\cancel{\vec{T}_{\phi\phi} = (-\sin\phi\sin\theta, -\sin\phi}$

$\vec{T}_{\phi\phi} = (-\cos\phi\cos\theta, -\cos\phi\sin\theta, -\sin\phi)$

$T_{\theta\phi} = (\sin\phi\sin\theta, -\sin\phi\cos\theta, 0)$

$l = -(R+\cos\phi)\cos\phi \quad m = 0 \quad n = -1$

$K = \frac{ln - m^2}{W} = \frac{(R+\cos\phi)\cos\phi}{(R+\cos\phi)^2} = \frac{\cos\phi}{R+\cos\phi}$

$\frac{1}{2\pi}\iint K\,dA =$

$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos\phi}{R+\cos\phi}\, R+\cos\phi \; d\theta\, d\phi =$

$\int_0^{2\pi} \cos\phi\, d\phi = \boxed{0} \rightarrow$ Agrees with Gauss-Bonnet theorem