DARIUSZ SIERGIEJUK 7.6 SURFACE INTEGRALS OF VECTOR FIELDS HOMEWORK #6

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1. Consider the closed surface S consisting of the graph $z=1-x^2-y^2$ with $z \geq 0$, and also the unit disc in the xy plane. Give this surface an outer normal. Compute:

$$\iint_S F \cdot dS$$

where $F(x,y,z)=(2x, 2y, z)$.

Consider the vector field F defined on S, where S is $z=g(x,y)$. The surface integral of F over S is

$$\iint_S F \cdot dS = \iint_D F \cdot (T_x \times T_y)\, dx\, dy$$

We have here $F(x,y,z)=(2x,2y,z)$ and surface S $z=1-x^2-y^2$ is the closed surface with $z \geqslant 0$ and $x^2+y^2 \leq 1$

Applying the parametrization

$x=u,\ y=v,\ z=1-u^2-v^2$

Obtaining T_u and T_v

$T_u = i - 2uk$

$T_v = j - 2vk$

so,

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = (0\cdot(-2v)+2u)i + (0\cdot(-2u)-(-2v))j + (1-0)k$$

$$= 2ui + 2vj + k$$

1. CONTINUATION

Next,

$$F\cdot(T_u \times T_v) = (2ui + 2vj + (1-u^2-v^2)k)\cdot(2ui+2vj+k)$$
$$= 4u^2+4v^2+1-u^2-v^2$$
$$= 3u^2+3v^2+1$$

The surface integral is,

$$\iint_S F\cdot dS = \iint_D F\cdot(T_u\times T_v)\,du\,dv$$
$$= \iint_D (3u^2+3v^2+1)\,du\,dv$$

Using polar coordinates to evaluate this integral.

$u = r\cos\theta$, $v = r\sin\theta$, $r = u^2+v^2$ and $dS = r\,dr\,d\theta$

So, the integral is,

$$\iint_S F\cdot dS = \int_0^{2\pi}\int_0^1 (3r^2+1)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (3r^3+r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{3r^4}{4}+\frac{r^2}{2}\right]_0^1 d\theta = [\theta]_0^{2\pi}\left(\frac{3}{4}+\frac{1}{2}\right) = (2\pi)\left(\frac{5}{4}\right) = \frac{5\pi}{2}$$

This way, the surface integral $\iint_S F\cdot dS$ equals $\frac{5\pi}{2}$.

4. let $F(x,y,z) = 2xi - 2yj + z^2k$. Evaluate

$$\iint_S F \cdot ds,$$

where S is the cylinder $x^2+y^2=4$ with $z \in [0,1]$.

In the oriented surface each point contains two unit vectors. Including one positive vector and another one negative.

The vector $u = xi + yj + zk$

The surface $\iint_S F \cdot u\, dS = 2A(S)$

$$A(S) = \int_0^2 \int_0^{2\pi} (F \cdot u)\, r\, dr\, d\theta$$

let the equation of sphere be $F(x,y,z) = 2xi - 2yj + z^2k$, this is also the oriented vector of the sphere point and the equation of the cylinder is $x^2+y^2=4$.

Differential

$T = x^2+y^2-4$

$\nabla T = 2x + 2y$

$\nabla T(x,y,z) = 2xi + 2yj$

Then, $\iint_S F \cdot u\, dS = 4A(S)$

$$A(S) = \iint \sqrt{\left(\frac{\delta z}{\delta x}\right)^2 + \left(\frac{\delta z}{\delta y}\right)^2 + 1} = \iint \sqrt{4x^2+4y^2+1}$$

4. CONTINUATION

$$= \iint \sqrt{4r^2+1}$$

$$A(S) = \int_0^2 \int_0^{2\pi} \sqrt{4r^2+1}\, r\, dr\, d\theta$$

Let $4r^2+1 = t \Rightarrow 8r\,dr = dt \Rightarrow r\,dr = \frac{1}{8}dt$

So the area of the cylinder is:

$$A(S) = \int_0^2 \int_0^{2\pi} \sqrt{4r^2+1}\, r\, dr\, d\theta = \frac{1}{8}\int_0^2 \int_0^{2\pi} \sqrt{t}\, dt\, d\theta$$

$$= \frac{2\pi}{8}\int_0^2 \sqrt{t}\, dt = \frac{\pi}{4}\left[\frac{2}{3}(t)^{\frac{3}{2}}\right]_0^2 = \frac{\pi}{6}\left[(4r^2+1)^{\frac{3}{2}}\right]_0^2 = \frac{\pi}{6}(69)$$

$$= 12\pi$$

Now,

$$\iint_S F \cdot n\, dS = 4(12\pi)$$

Therefore, the surface integral of the cylinder equals 48π.

5. Let the temperature of a point in $\mathbb{R}^3$ be given by $T(x,y,z) = 3x^2+3z^2$.
Compute the heat flux across the surface $x^2+z^2=2$, $0 \le y \le 2$, if $k=1$.

The surface integral $\iint_S F \cdot dS$, taken as a normal component of F, becomes surface integral,

$$\iint_S F \cdot dS = \iint_S F \cdot n \, dS$$

The value of F when temperature $T(x,y,z)$ is, $F = -k\nabla T$
here k is the positive constant.
The heat flux across the surface S, with $k=1$, will be

$$\int_S F \cdot dS = -\int_S \nabla T \cdot dS$$

The temperature of a point $T(x,y,z) = 3x^2+3z^2$.
The surface is the radius of the cylinder $\sqrt{2}$ with axis of symmetry located along the y-axis with top and bottom taken away.

Using cylindrical coordinates to parametrize S;

$$\phi(u,v) = (\sqrt{2}\cos u, v, \sqrt{2}\sin u), \quad (u,v) \in [0,2\pi] \times [0,2]$$

Next,

$$T_u = (-\sqrt{2}\sin u, 0, \sqrt{2}\cos u)$$

$$T_v = (0, 1, 0)$$

5. CONTINUATION

This way,

$$n(u,v) = T_u \times T_v$$
$$= (-\sqrt{2}\sin u, 0, \sqrt{2}\cos u) \times (0,1,0)$$
$$= (-\sqrt{2}\cos u, 0, -\sqrt{2}\sin u)$$

Aside:

$$T_u \times T_v \begin{vmatrix} i & j & k \\ -\sqrt{2}\sin u & 0 & \sqrt{2}\cos u \\ 0 & 1 & 0 \end{vmatrix}$$
$$= \sqrt{2}\cos u, 0, -\sqrt{2}\sin u$$

and

$$\nabla T(x,y,z) = (6x, 0, 6z)$$
$$\nabla T(u,v) = (6\sqrt{2}\cos u, 0, 6\sqrt{2}\sin u)$$

Therefore, the flux is

$$\int_S F \cdot dS = \int_S -\nabla T \cdot dS = \int_S (-6x, 0, -6z) \cdot dS$$

$$= \int_0^2 \int_0^{2\pi} (-6\sqrt{2}\cos u, 0, -6\sqrt{2}\sin u) \cdot (-\sqrt{2}\cos u, 0, -\sqrt{2}\sin u)\, du\, dv$$

$$= \int_0^2 \int_0^{2\pi} (12\cos^2 u + 12\sin^2 u)\, du\, dv = \int_0^2 \int_0^{2\pi} 12\, du\, dv = 12\int_0^2 [u]_0^{2\pi}\, dv$$

$$= 24\pi [v]_0^2 = 24\pi [v]_0^2 = 48\pi$$

Hence, the heat flux across the given surface S is 48π.

6. Compute the heat flux across the unit sphere S if $T(x,y,z)=x$. Can you interpret your answer physically?

The surface integral $\iint_S F\cdot dS$, taken as a normal component of F, becomes surface integral,

$$\iint_S F\cdot dS=\iint_S F\cdot n\,dS$$

The value of F when temperature $T(x,y,z)$ is, $F=-k\nabla T$ here k is the positive constant.

Here, we have the temperature of a point $T(x,y,z)=x$.
Evaluate the heat flux across the unit sphere, if $k=1$.

The value of F is,

$$F=-k\nabla T=-\left(\frac{\partial}{\partial x}(x)\right)i=-i$$

The outward unit normal to the sphere is,

$$n=\frac{r}{\|r\|}=\frac{xi+yj+zk}{1}$$

Aside: Radius is 1

$$=xi+yj+zk$$

Next,

$$F\cdot n=(-i)\cdot(xi+yj+zk)=-x$$

Compute the heat flux as shown below:

$$\iint_S F\cdot dS=\iint_S F\cdot n\,dS=-\iint_S x\,dS$$

6. CONTINUATION

Now, S is a unit sphere.

This way, $y = \pm\sqrt{1-x^2}$ and $x = \pm 1$

Next,

$$\iint_S F \cdot dS = -\iint_S x\, dS = -\iint_S x\, dx\, dy$$

Use polar coordinates to evaluate the integral value.

let $x = r\cos\theta$ $\quad dx\,dy = r\,dr\,d\theta$

$$\left.\begin{aligned} \theta &: 0 \le \theta \le 2\pi \\ r &: 0 \le r \le 1 \end{aligned}\right\} \text{ since S is unit sphere.}$$

$$\iint_S F \cdot dS = -\int_0^1 \int_0^{2\pi} r^2 \cos\theta\, dr\, d\theta = -\int_0^1 r^2\, dr \times \int_0^{2\pi} \cos\theta\, d\theta$$

$= 0$

Aside:

$$\int_0^{2\pi} \cos\theta\, d\theta = 0 \qquad \sin\theta\Big|_0^{2\pi} = 0$$

Therefore, the flux across unit sphere for the function is zero.

9. Evaluate $\iint_S (\nabla \times F) \cdot dS$, where S is the surface $x^2+y^2+3z^2=1$, $z \leq 0$ and F is the vector field $F = yi - xj + zx^3y^2k$.
(Let n, the unit normal, be upward pointing.)

S is the surface $x^2+y^2+3z^2=1$, $z \leq 0$
and the following vector field: $F(x,y,z) = yi - xj + zx^3y^2k$

Our goal is to evaluate $\iint_S (\nabla \times F) \cdot dS$.

$\nabla \times F$ can be calculated as follows:

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & zx^3y^2 \end{vmatrix} = (2yzx^3 - 0)i - (3x^2zy^2 - 0)j + (-1-1)k$$

$$= 2yzx^3 i - 3x^2zy^2 j - 2k$$

$$z = g(x,y) = -\sqrt{\frac{1-x^2-y^2}{3}}$$

Aside:

$$x^2+y^2+3z^2=1$$

$$3z^2 = 1-x^2-y^2$$

$$z = \sqrt{\frac{1-x^2-y^2}{3}}$$

The surface integral of a vector field over a graph S is given by,

$$\iint_S F \cdot ds = \iint_D F \cdot (T_x \times T_y)\, dS$$

$$= \iint_D \left(F_1\left(-\frac{\partial g}{\partial x}\right) + F_2\left(-\frac{\partial g}{\partial y}\right) + F_3 \right) dx\,dy$$

9. CONTINUATION

Calculating the partial differentials of $g(x,y)$

$$\frac{\partial g(x,y)}{\partial x} = -\frac{\partial}{\partial x}\left(\sqrt{\frac{1-x^2-y^2}{3}}\right)$$
$$= -\frac{1}{\sqrt{3}}\frac{\partial}{\partial x}\left(\sqrt{1-x^2-y^2}\right)$$
$$= -\frac{-2x}{2\sqrt{3}\sqrt{1-x^2-y^2}}$$
$$= \frac{x}{\sqrt{3}\sqrt{1-x^2-y^2}}$$

Aside: Applying $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$, $u=f(x)$

$$\frac{\partial g(x,y)}{\partial y} = -\frac{\partial}{\partial y}\left(\sqrt{\frac{1-x^2-y^2}{3}}\right)$$
$$= -\frac{1}{\sqrt{3}}\frac{\partial}{\partial y}\left(\sqrt{1-x^2-y^2}\right)$$
$$= -\frac{-2y}{2\sqrt{3}\sqrt{1-x^2-y^2}}$$
$$= \frac{y}{\sqrt{3}\sqrt{1-x^2-y^2}}$$

Aside: Applying $\frac{d}{dy}(u^n) = nu^{n-1}\frac{du}{dy}$, $u=f(y)$

This way, the surface integral $\iint_S (\nabla\times F)\cdot dS$ can be calculated as follows:

$$\iint_S (\nabla\times F)\cdot dS = \iint_D (\nabla\times F)\cdot(T_x\times T_y)\,dS$$
$$= \iint_D \left(F_1\left(-\frac{\partial g}{\partial x}\right) + F_2\left(-\frac{\partial g}{\partial y}\right) + F_3\right)dx\,dy$$

9. CONTINUATION

$$=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\left(-\frac{2y\cancel{z}x^4}{\sqrt{3}\sqrt{1-x^2-y^2}}+\frac{\sqrt{3}x^2y^3z}{\sqrt{1-x^2-y^2}}-2\right)dx\,dy$$

Aside: Substituting $z=-\sqrt{\frac{1-x^2-y^2}{3}}$

$$=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\left(\frac{2yx^4}{3}-x^2y^3-2\right)dx\,dy$$

Aside: Applying $\int x^u dx=\frac{x^{u+1}}{u+1}+C$

$$=\int_{-1}^{1}\left[\frac{2yx^5}{15}-\frac{x^3y^3}{3}-2x\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}dy$$

$$=\int_{-1}^{1}\left(\frac{4y(1-y^2)^{5/2}}{15}-\frac{2y^3(1-y^2)^{3/3}}{3}-4\sqrt{1-y^2}\right)dy$$

Evaluating the integral $\int_{-1}^{1}\frac{4(1-y^2)^{5/2}}{15}dy$

Letting $y=\cos\theta$

$dy=-\sin\theta\, d\theta$ where $\pi\leq\theta\leq 1$

Next, $\int_{-1}^{1}\frac{4y(4-y^2)^{5/2}}{15}dy=\frac{-4}{15}\int_{\pi}^{0}\cos\theta\sin^6\theta$

$$=\frac{4}{15}\int_{0}^{\pi}\cos\theta\sin^6\theta=0$$

Aside:
$\cos(\pi)=-1$
$\cos(0)=1$
$\sin(\pi)=0$
$\sin(0)=0$

9. CONTINUATION

Evaluating the integral $\int_{-1}^{1} \frac{2y^3(1-y^2)^{3/2}}{3} dy$

Letting $y = \cos\theta$

$dy = -\sin\theta d\theta$ where $\pi \le \theta \le 1$

Next, $\int_{-1}^{1} \frac{2y^3(1-y^2)^{3/2}}{3} dy = \frac{2}{3}\int_{\pi}^{0} \cos^3\theta \sin^3\theta(-\sin\theta)d\theta$

$= \frac{2}{3}\int_{0}^{\pi} \cos^3\theta \sin^4\theta d\theta = 0$

Aside: $1-\cos^2\theta = \sin^2\theta$

Aside: $\sin(\pi)=0$ $\sin(0)=0$

Evaluating the integral $\int_{-1}^{1} 4\sqrt{1-y^2}dy$

Letting $y = \cos\theta$

$dy = -\sin\theta d\theta$ wher $\pi \le \theta \le 1$

Next, $\int_{-1}^{1} 4\sqrt{1-y^2}dy = 4\int_{\pi}^{0} \sin\theta(-\sin\theta)d\theta$

$= -4\int_{\pi}^{0} \sin^2\theta d\theta$

$= -4\int_{\pi}^{0} \frac{1-\cos 2\theta}{2} d\theta$

$= -2\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_{\pi}^{0} = 2\pi$

Now plugging the obtained values into $\int_{-1}^{1}\left(\frac{4y(1-y^2)^{5/2}}{15} - \frac{2y^3(1-y^2)^{3/2}}{3} - 4\sqrt{1-y^2}\right)dy$

$\iint_S (\nabla \times F)\cdot dS = 0 - 0 - 2\pi = -2\pi$

Hence, the value of $\iint_S (\nabla \times F)\cdot dS$ is $\underline{-2\pi}$.

17. Work out a formula like that in Exercise 15 for integration over the surface of a cylinder.

The surface integral of a vector field F over S is

$$\iint_S F\cdot dS = \iint_D F\cdot(T_\theta \times T_z)\, d\theta\, dz$$

Consider a surface, S a cylinder with unit radius and a vector field F.

The parametrization for the cylinder is

$$\phi(\theta, z) = (\cos\theta, \sin\theta, z)$$

The value of T_θ and T_z is

$$T_\theta = (-\sin\theta)i + (\cos\theta)j$$

$$T_z = 1k$$

Aside:

$$T_\theta \times T_z \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta, -(-\sin\theta)$$

The vector cross product is $T_\theta \times T_z = (\cos\theta)i + (\sin\theta)j = r$

Therefore, the surface integral is,

$$\iint_S F\cdot ds = \iint_D F\cdot(T_\theta \times T_z)\, d\theta\, dz$$

$$= \iint_D (F\cdot r)\, d\theta\, dz$$

$$= \int_{z=a}^{z=b}\int_{\theta=0}^{\theta=2\pi} F_r\, d\theta\, dz$$

22. If S is the upper hemisphere $\{(x,y,z) \mid x^2+y^2+z^2=1, z \geq 0\}$ oriented by the normal pointing out of the sphere, compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for parts (a) and (b).

(a) $F(x,y,z) = xi + yj$

(b) $F(x,y,z) = yi + xj$

(c) For each of these vector fields, compute

$\iint_S (\nabla \times F) \cdot dS$ and $\int_C F \cdot ds$, where C is the unit circle in the xy plane traversed in the counterclockwise direction (as viewed from the positive z axis).

(Notice that C is the boundary of S. The phenomenon illustrated here will be studied more thoroughly in the next chapter, using Stokes' theorem.)

(a) Considering the following vector field and surface of upper hemisphere:

$$F(x,y,z) = xi + yj$$

$$S = \{(x,y,z) \mid x^2+y^2+z^2 = 1, z \geq 0\}$$

Let the surface be represented as follows:

$$z = g(x,y) = \sqrt{1-x^2-y^2}$$

22. CONTINUATION

The surface integral of a vector field over a graph S is,

$$\iint_S F\cdot dS = \iint_D F\cdot(T_x \times T_y)\,dS$$

$$= \iint_D \left(F_1\left(-\frac{\partial g}{\partial x}\right) + F_2\left(-\frac{\partial g}{\partial y}\right) + F_3\right)dx\,dy \quad \checkmark$$

Calculating the partial differentials of $g(x,y)$

$$\frac{\partial g(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{1-x^2-y^2}\right)$$

$$= \frac{-2x}{2\sqrt{1-x^2-y^2}} = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$\frac{\partial g(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{1-x^2-y^2}\right)$$

$$= \frac{-2y}{2\sqrt{1-x^2-y^2}} = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Computing the surface integral(s)

$$\iint_S F\cdot dS = \iint_D F\cdot(T_x \times T_y)\,dS$$

$$= \iint_D \left(F_1\left(-\frac{\partial g}{\partial x}\right) + F_2\left(-\frac{\partial g}{\partial y}\right) + F_3\right)dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{x^2}{\sqrt{1-x^2-y^2}} + \frac{y^2}{\sqrt{1-x^2-y^2}}\right)dx\,dy$$

22. CONTINUATION

$$=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\left(\frac{x^2+y^2}{\sqrt{1-x^2-y^2}}\right)dx\,dy$$

Let $x=r\cos\theta$, $y=r\sin\theta$

$dx\,dy=r\,dr\,d\theta$

S is the upper hemisphere $x^2+y^2+z^2=1$

So, $0\le r\le 1$, $0\le\theta\le 2\pi$

Next,

$$\iint_S F\cdot dS=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\left(\frac{x^2+y^2}{\sqrt{1-x^2-y^2}}\right)dx\,dy$$

$$=\int_0^{2\pi}\int_0^1\frac{r^3}{\sqrt{1-r^2}}\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta * \int_0^1\frac{r^3}{\sqrt{1-r^2}}\,dr=2\pi\Big[\underbrace{-\frac{1}{3}\sqrt{1-r^2}(r^2+2)}_{2/3}\Big]_0^1$$

$$=2\pi\left(\frac{2}{3}\right)=\frac{4\pi}{3}.$$

The value of $\iint_S F\cdot dS$ is $\frac{4\pi}{3}$.

(b) Consider the vector field $F(x,y,z)=yi+xj$ across the upper hemisphere,

$S(x,y,z)\,|\,x^2+y^2+z^2=1,\ z\geq 0$

22. CONTINUATION

Let the surface be represented as,

$$z = g(x,y) = \sqrt{1-x^2-y^2}$$

The surface integral is,

$$\iint_S F \cdot dS = \iint_D F \cdot (T_x \times T_y) dS$$

$$= \iint_D \left(F_1 \left(\frac{-\partial g}{\partial x} \right) + F_2 \left(\frac{-\partial g}{\partial y} \right) + F_3 \right) dx\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{yx}{\sqrt{1-x^2-y^2}} + \frac{xy}{\sqrt{1-x^2-y^2}} \right) dx\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{2xy}{\sqrt{1-x^2-y^2}} \right) dx\,dy$$

We can evaluate the integral in the following way:

Let $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

S is upper hemisphere $x^2+y^2+z^2=1$

Then, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

We have, $\displaystyle \iint_S F \cdot dS = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{2xy}{\sqrt{1-x^2-y^2}} \right) dx\,dy$

$$= \int_0^{2\pi} \int_0^1 \left(\frac{2r^3 \sin\theta\cos\theta}{\sqrt{1-r^2}} \right) dr\,d\theta$$

22. CONTINUATION

$$= \int_0^1 \frac{r^3}{\sqrt{1-r^2}}\, dr \times \left(\frac{-\cos 2\theta}{2}\right)_0^{2\pi} = 0$$

Aside: $\cos(4\pi) = \cos 0 = 1$

Therefore, the value of $\iint_S F \cdot dS$ is $\underline{\underline{0}}$.

(c) Let's consider that C is the unit circle in xy-plane for both vector fields $F(x,y,z) = xi + yj$ and $F(x,y,z) = yi + xj$.

$\nabla \times F$ for both vector fields is

$\nabla \times F = 0$

$$\iint_S (\nabla \times F) \cdot dS = \iint_S 0\, dS = 0$$

The unit normal to surface C is,

$n = k$

The scalar product with both vector fields is,

$F \cdot n = (xi + yj) \cdot (k) = 0$

$F \cdot n = (yi + xj) \cdot (k) = 0$

The integral becomes,

$$\int_C F \cdot ds = \int_C F \cdot n\, ds = \int_C 0\, dS = 0$$

Therefore, the value of $\iint_S (\nabla \times F) \cdot dS$ and $\int_C F \cdot ds$ is $\underline{\underline{0}}$ for both vector fields.

4. The helicoid can be described by

$$\Phi(u,v)=(u\cos v, u\sin v, bv), \text{ where } b\neq 0.$$

Show that $H=0$ and that $K=-b^2/(b^2+u^2)^2$. In Figures 7.7.1 and 7.7.5, we see that the helicoid is actually a soap film surface. Surfaces in which $H=0$ are called minimal surfaces.

Consider that $\Phi: D\longrightarrow \mathbb{R}^3$ is a smooth parametrized surface and tangent vectors to this surface are,

$$T_u=\frac{\partial\Phi}{\partial u}, \quad T_v=\frac{\partial\Phi}{\partial v}$$

Some terms here are,

$$E=\left\|\frac{\partial\Phi}{\partial u}\right\|^2 \qquad F=\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial v} \qquad G=\left\|\frac{\partial\Phi}{\partial v}\right\|^2$$

$$\|T_u\times T_v\|^2=W=EG-F^2$$

Let

$$N=\frac{T_u\times T_v}{\|T_u\times T_v\|}=\frac{T_u\times T_v}{\sqrt{W}}$$

Three additional functions are defined

$$l(p)=N(u,v)\cdot\phi_{uu}$$

$$m(p)=N(u,v)\cdot\phi_{uv}$$

$$n(p)=N(u,v)\cdot\phi_{vv}$$

The Gauss Curvature at point p is $K(p)=\dfrac{ln-m^2}{W}$

The Mean Curvature at point p is

$$H(p) = \frac{Gl + En - 2Fm}{2W}$$

Consider the helicoid described by $\Phi(u,v) = (u\cos v, u\sin v, bv)$ where $b \neq 0$

The tangent vectors are,

$$T_u = \frac{\partial \Phi}{\partial u} = \cos v\, i + \sin v\, j$$

$$T_v = \frac{\partial \Phi}{\partial v} = -u\sin v\, i + u\cos v\, j + bk$$

Some terms,

$$E = \left\| \frac{\partial \Phi}{\partial u} \right\|^2 = (1 + (u')^2)$$

$$F = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} = 0$$

Aside:

$(\cos v\, i + \sin v\, j + 0) \cdot$
$(-u\sin v\, i + u\cos v\, j + bk)$
$-u\cos v\sin v + u\cos v\sin v + 0$
$= \underline{\underline{0}}.$

$$G = \left\| \frac{\partial \Phi}{\partial v} \right\|^2 = u^2 + b^2$$

And,

$$N = \frac{T_u \times T_v}{\| T_u \times T_v \|} = \frac{T_u \times T_v}{\sqrt{W}}$$

$$= \frac{1}{\sqrt{u^2 + b^2}} (b\sin v\, i - b\cos v\, j + uk)$$

1. CONTINUATION

There are three more functions as here,

$$l(p) = N(u,v) \cdot \Phi_{uu} = 0$$

$$m(p) = N(u,v) \cdot \Phi_{uv} = \frac{-b}{\sqrt{u^2+b^2}}$$

$$n(p) = N(u,v) \cdot \Phi_{vv} = \frac{-h}{\sqrt{1+(h')^2}}$$

The Gauss Curvature and Mean Curvature is

$$K(p) = \frac{ln - m^2}{W} = \frac{-b^2}{(b^2-u^2)^2}$$

$$H(p) = \frac{Gl - En - 2Fm}{2W} = \boxed{0}$$

6. Compute the Gauss curvature of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1.$$

Consider $\Phi: D \mapsto \mathbb{R}^3$ a smooth parametrized surface and tangent vectors to this surface are,

$$T_u=\frac{\partial \Phi}{\partial u}, \quad T_v=\frac{\partial \Phi}{\partial v}$$

Additional terms,

$$E=\left\|\frac{\partial \Phi}{\partial u}\right\|^2 \quad F=\frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial v} \quad G=\left\|\frac{\partial \Phi}{\partial v}\right\|^2$$

$$\|T_u \times T_v\|^2 = W = EG-F^2$$

And,

$$N=\frac{T_u \times T_v}{\|T_u \times T_v\|}=\frac{T_u \times T_v}{\sqrt{W}}$$

Three more functions are as below,

$$l(p)=N(u,v)\cdot \Phi_{uu}$$
$$m(p)=N(u,v)\cdot \Phi_{uv}$$
$$n(p)=N(u,v)\cdot \Phi_{vv}$$

The Gauss Curvature at point p is $K(p)=\dfrac{ln-m^2}{W}$

The Mean Curvature at point p is $H(p)=\dfrac{Gl+En-2Fm}{2W}$

Consider the ellipsoid described by

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1$$

6. CONTINUATION

Parametrizing the surface with $\Phi(u,v)$ as

$\Phi(u,v) = (a\cos u\sin v, a\sin u\sin v, c\cos v)$

The tangent vectors are,

$$T_u = \frac{\partial \Phi}{\partial u} = -a(\sin v\sin u)i + a(\sin v\cos u)j$$

$$T_v = \frac{\partial \Phi}{\partial v} = a\cos u\cos v\, i + a\sin u\cos v\, j - c\sin v\, k$$

The terms are

$$E = \left\|\frac{\partial \Phi}{\partial u}\right\|^2 = a^2\sin^2 v$$

Aside:

$$T_u = -a(\sin v\sin u)i + a(\sin v\cos u)j$$

$$(T_u)^2 = (-a(\sin v\sin u))^2 + (a(\sin v\cos u))^2$$

$$= a^2\sin^2 v\sin^2 u + a^2\sin^2 v\cos^2 u$$

$$= a^2\sin^2 v(\sin^2 u + \cos^2 u)$$

$$= a^2\sin^2 v$$

$$F = \frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial v} = 0$$

Aside:

$$\frac{\partial \Phi}{\partial u} = -a\sin v\sin u + a\sin v\cos u + 0$$

$$\frac{\partial \Phi}{\partial v} = a\cos u\cos v + a\sin u\cos v - c\sin v$$

$$= \underbrace{-a^2\sin u\cos u\sin v\cos v + a^2\sin u\cos u\sin v\cos v}_{0} + 0$$

6. CONTINUATION

$$G = \left\|\frac{\partial\Phi}{\partial v}\right\|^2 = a^2\cos^2 v + c^2\sin^2 v$$

Aside:

$$\frac{\partial\Phi}{\partial v} = a\cos u\cos v + a\sin u\cos v - c\sin v$$

$$\left[\frac{\partial\Phi}{\partial v}\right]^2 = \underline{a^2}\cos^2 u\,\underline{\cos^2 v} + \underline{a^2}\sin^2 u\,\underline{\cos^2 v} + c^2\sin^2 v$$

$$= a^2\cos^2 v(\cos^2 u + \sin^2 u) + c^2\sin^2 v$$

$$= a^2\cos^2 v + c^2\sin^2 v$$

$$\|T_u \times T_v\|^2 = W = EG - F^2$$

$$= \underbrace{a^2\sin^2 v}_{E}\underbrace{(a^2\cos^2 v - c^2\sin^2 v)}_{G} - \underbrace{0^2}_{F}$$

$$= a^4\sin^2 v\cos^2 v - a^2c^2\sin^4 v$$

And,

$$N = \frac{T_u \times T_v}{\|T_u \times T_v\|} = \frac{T_u \times T_v}{\sqrt{W}}$$

$$= \frac{1}{a\sin v\sqrt{(a^2\cos^2 v + c^2\sin^2 v}} \cdot (-ac\sin^2 v\cos u\, i - ac\sin u\sin^2 v\, j - a^2\sin v\cos v\, k)$$

Aside:

$$a^4\sin^2 v\cos^2 v + a^2c^2\sin^4 v$$
$$a^2\sin^2 v(a^2\cos^2 v + c^2\sin^2 v)$$

Three more functions are as here,

$$L(p) = N(u,v)\cdot\Phi_{uu} = \frac{a^2c\sin^3 v}{a\sin v\sqrt{(a^2\cos^2 v + c^2\sin^2 v}}$$

6. CONTINUATION

$$m(p) = N(u,v)\cdot \Phi_{uv} = 0$$

$$n(p) = N(u,v)\cdot \Phi_{vv} = \frac{a^2 c \sin v}{a \sin v \sqrt{(a^2\cos^2 v + c^2 \sin^2 v)}}$$

Therefore, the Gauss Curvature is

$$K(p) = \frac{ln - m^2}{W}$$

$$= \frac{\left(\dfrac{a^2 c \sin^3 v}{a \sin v \sqrt{(a^2\cos^2 v + c^2\sin^2 v)}}\right)\left(\dfrac{a^2 c \sin v}{a \sin v \sqrt{(a^2\cos^2 v + c^2 \sin^2 v)}}\right)}{a^4 \sin^4 v \cos^2 v + a^2 c^2 \sin^4 v}$$

$$= \frac{a^4 c^2 \sin^4 v}{(a^4 \sin^2 v \cos^2 v + a^2 c^2 \sin^4 v)^2}$$

$$= \frac{c^2}{(a^2\cos^2 v + c^2 \sin^2 v)^2}$$

Therefore, the Gauss Curvature of ellipsoid

$\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} + \dfrac{z^2}{c^2} = 1$ is equal to

$$\frac{c^2}{(a^2\cos^2 v + c^2\sin^2 v)^2}$$

7. After finding K in Exercise 6, integrate K to show that:

$$\frac{1}{2\pi}\iint_S K\,dA = 2.$$

We have this ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1$$

The Gauss curvature of the ellipsoid is

$$K=\frac{a^4c^2}{(a^4\cos^2 v+a^2c^2\sin^2 v)^2}=\frac{a^4c^2}{a^4(a^2\cos^2 v+c^2\sin^2 v)^2}=\frac{c^2}{(a^2\cos^2 v+c^2\sin^2 v)^2}$$

The surface element for the ellipsoid,

$$\|T_u\times T_v\|=\sin v\sqrt{a^4\cos^2 v+a^2c^2\sin^2 v}$$

The integral,

$$\frac{1}{2\pi}\iint_S K\,dA=\frac{1}{2\pi}\int_0^{\pi}\int_0^{2\pi}K\cdot(T_u\times T_v)\,du\,dv$$

$$=\frac{1}{2\pi}\int_0^{\pi}\int_0^{2\pi}\frac{a^4c^2\sin v}{((a^4\cos^2 v+a^2c^2\sin^2 v))^{3/2}}\,du\,dv$$

$$=\int_0^{\pi}\frac{a^4c^2\sin v}{((a^4\cos^2 v+a^2c^2\sin^2 v))^{3/2}}\,dv$$

$$=ac^2\int_0^{\pi}\frac{\sin v}{((a^2\cos^2 v+c^2(1-\cos^2 v)))^{3/2}}\,dv$$

7. CONTINUATION

$$= \frac{ac^2}{(a^2-c^2)^{3/2}} \int_0^{\pi} \frac{\sin v}{\left(\cos^2 v + \frac{c^2}{(a^2-c^2)}\right)^{3/2}} dv$$

substituting $w = \cos v$

$$\frac{1}{2\pi}\iint_S K\,dA = \frac{ac^2}{(a^2-c^2)^{3/2}} \int_{-1}^{1} \frac{1}{\left((w)^2 + \left(\sqrt{\frac{c^2}{(a^2-c^2)}}\right)^2\right)^{3/2}} dv$$

Now, plugging in our $w = \sqrt{\frac{c^2}{(a^2-c^2)}} \tan\theta$

$$\frac{1}{2\pi}\iint_S K\,dA = \frac{ac^2}{(a^2-c^2)^{3/2}} \int \frac{\sec^2\theta}{\frac{c^2}{(a^2-c^2)} (\sec^2\theta)^{3/2}} d\theta$$

$$= \frac{a}{(a^2-c^2)^{1/2}} \left[\sqrt{\frac{w^2(a^2-c^2)}{c^2+w^2(a^2-c^2)}}\right]_{-1}^{1} = \underline{\underline{2.}}$$

9. Show that Enneper's surface

$$\Phi(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\right)$$

is a minimal surface ($H=0$).

Enneper's surface is described by the above.

Calculating the tangent vectors first,

$$T_u = \frac{\partial \Phi}{\partial u} = (1 - u^2 + v^2)\hat{i} + 2uv\hat{j} + 2u\hat{k}$$

$$T_v = \frac{\partial \Phi}{\partial v} = 2uv\hat{i} + (1 + v^2 + u^2)\hat{j} - 2v\hat{k}$$

Our terms are as follows,

$$E = \left\| \frac{\partial \Phi}{\partial u} \right\|^2 = (v^2 + u^2 + 1)^2$$

$$F = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} = 0$$

$$G = \left\| \frac{\partial \Phi}{\partial v} \right\|^2 = (v^2 + u^2 + 1)^2$$

$$\|T_u \times T_v\|^2 = W = EG = F^2 = (v^2 + u^2 + 1)^4$$

$$N = \frac{T_u \times T_v}{\|T_u \times T_v\|} = \frac{T_u \times T_v}{\sqrt{W}}$$

$$= \frac{1}{(1 + u^2 + v^2)^2} \left(-2u(1 + v^2 + u^2)\hat{i} + 2v(1 + u^2 + v^2)\hat{j} + (1 - (v^2 + u^2)^2)\hat{k}\right)$$

9. CONTINUATION

Additional three functions are as follows,

$$l(p) = N(u,v) \cdot \Phi_{uu} = 1$$

$$m(p) = N(u,v) \cdot \Phi_{uv} = 0$$

$$n(p) = N(u,v) \cdot \Phi_{vv} = -1$$

The Mean curvature is,

$$H(p) = \frac{Gl + En - 2Fm}{2W}$$

$$H(p) = \frac{(1+u^2+v^2)^2(1) - (1)(1+u^2+v^2)^2 - 0}{2(1+u^2+v^2)4}$$

$$H(p) = 0$$

Therefore, the Enneper's surface is going to be a minimal surface.

10. Consider the torus T given in Exercise 4, Section 7.4. Compute its Gauss curvature and verify the theorem of Gauss-Bonnet. [Hint: Show that $\|T_\theta \times T_\phi\|^2 = (R+\cos\phi)^2$ and $K = \cos\phi / (R+\cos\phi)$.]

Our parametrization of this torus,

$\Phi(\theta, \phi) = ((R+\cos\phi)\cos\theta, (R+\cos\phi)\sin\theta, \sin\phi)$

where $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq 2\pi$. The limits for integral.

First computing our tangent vectors,

$$T_\theta = \frac{\partial \Phi}{\partial \theta} = -(R+\cos\phi)\sin\theta\,\hat{i} + (R+\cos\phi)\cos\theta\,\hat{j}$$

$$T_\phi = \frac{\partial \Phi}{\partial \phi} = -(\sin\phi\cos\theta)\hat{i} - (\sin\phi\sin\theta)\hat{j} + \cos\phi\,\hat{k}$$

Calculating some required terms here,

$$E = \left\|\frac{\partial \Phi}{\partial \theta}\right\|^2 = (R+\cos\phi)^2$$

$$F = \frac{\partial \Phi}{\partial \theta} \cdot \frac{\partial \Phi}{\partial \phi} = 0$$

$$G = \left\|\frac{\partial \Phi}{\partial \phi}\right\|^2$$

$$\|T_\theta \times T_\phi\|^2 = W = EG - F^2 = (R+\cos\phi)^2$$

Plus,

$$N = \frac{T_\theta \times T_\phi}{\|T_\theta \times T_\phi\|} = \frac{T_\theta \times T_\phi}{\sqrt{W}}$$

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10. CONTINUATION

$$= \frac{1}{(R+\cos\phi)}\left((R+\cos\phi)\cos\phi\cos\theta\hat{i} + (R+\cos\phi)\cos\phi\sin\theta\hat{j} + (R+\cos\phi)\sin\phi\hat{k}\right)$$

Additional three functions are as follows here,

$$l(p) = N(\theta,\phi)\cdot\Phi_{\theta\theta} = -\cos\phi$$

$$m(p) = N(\theta,\phi)\cdot\Phi_{\theta\phi} = 0$$

$$n(p) = N(\theta,\phi)\cdot\Phi_{\phi\phi} = -1$$

The Gauss curvature formula,

$$K(p) = \frac{ln - m^2}{W} = \frac{(-\cos\phi)(-1)-(0)^2}{(R+\cos\phi)} = \underline{\underline{\frac{\cos\phi}{(R+\cos\phi)}}}.$$

The Gauss-Bonnet theorem is,

$$\frac{1}{2\pi}\iint_S K\,dA = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} K\cdot\|T_\theta \times T_\phi\|\,d\theta\,d\phi = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos\phi}{(R+\cos\phi)}(R+\cos\phi)\,d\theta\,d\phi$$

$$= \frac{2\pi}{2\pi}\int_0^{2\pi}\cos\phi\,d\phi.$$

Aside:

$$\int\cos x\,dx = \sin x + C$$

$$= [\sin\phi]_0^{2\pi}$$

$= 0$ Applying the limits for ϕ here.

$$= 2-2g \quad (g=1)$$

The Gauss-Bonnet theorem is verified for ~~torus~~ torus.