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HW 8
4/21/17

2/2 Excellent!

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Homework #7, Section 8.1 #6, 12, 14, 15, 19, 20, 24, 25, 27, 28

#6 $D = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, $P(x,y) = \sin x$, $Q(x,y) = \cos y$

$$\int_{\partial D} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\left.\begin{array}{l}\frac{\partial Q}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0\end{array}\right\} \iint_D (0-0)\,dx\,dy = 0 \checkmark$$

#12 Using the divergence theorem, show that $\int_{\partial D} \vec{F} \cdot \vec{n}\, ds = 0$, where $\vec{F}(x,y) = y\hat{i} - x\hat{j}$ and D is the unit disk.

$$\iint_D \nabla \cdot F\, dA = \iint_D \left(\frac{\partial}{\partial x}(y)\hat{i} - \frac{\partial}{\partial y}(x)\hat{j}\right) dA = 0$$

$$\therefore \int_{\partial D} \vec{F} \cdot \vec{n}\, ds = 0 \checkmark$$

#14 Under conditions of Green's theorem, prove that

(a) $\int_{\partial D} PQ\,dx + PQ\,dy = \iint_D \left[Q\left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right) + P\left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right)\right] dx\,dy$

$$\int_{\partial D} PQ\,dx + PQ\,dy$$

$$= \iint_D \frac{\partial}{\partial x}(PQ) - \frac{\partial}{\partial y}(PQ)\,dx\,dy$$

$$= \iint_D \left(\frac{\partial P}{\partial x}(Q) + \frac{\partial Q}{\partial x}(P)\right) - \left(\frac{\partial P}{\partial y}(Q) + \frac{\partial Q}{\partial y}\right) dx\,dy$$

$$= \iint_D \left[Q\left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right) + P\left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right)\right] dx\,dy = \int_{\partial D} PQ\,dx + PQ\,dy \checkmark$$

(b) $\int_{\partial D} \left(Q\frac{\partial P}{\partial x} - P\frac{\partial Q}{\partial x}\right) dx + \left(P\frac{\partial Q}{\partial y} - Q\frac{\partial P}{\partial y}\right) dy = 2\iint_D \left(P\frac{\partial^2 Q}{\partial x \partial y} - Q\frac{\partial^2 P}{\partial x \partial y}\right) dx\,dy$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(P\frac{\partial Q}{\partial y} - Q\frac{\partial P}{\partial y}\right) - \frac{\partial}{\partial y}\left(Q\frac{\partial P}{\partial x} - P\frac{\partial Q}{\partial x}\right)\right] dx\,dy$$

$$= \iint_D \left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial Q}{\partial y}\right) + P\left(\frac{\partial^2 Q}{\partial x \partial y}\right) - \left[\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial P}{\partial y}\right) + \left(Q\left(\frac{\partial^2 P}{\partial x \partial y}\right)\right)\right] - \left[\left(\frac{\partial Q}{\partial y}\right)\left(\frac{\partial P}{\partial x}\right) + Q\frac{\partial^2 P}{\partial x \partial y} - \left(\left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial Q}{\partial x}\right) + P\left(\frac{\partial^2 Q}{\partial y \partial x}\right)\right)\right] dx\,dy$$

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$$=\iint_D\left[\cancel{\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial Q}{\partial y}\right)}-\cancel{\left(\frac{\partial Q}{\partial y}\right)\left(\frac{\partial P}{\partial x}\right)}+\cancel{\left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial Q}{\partial x}\right)}-\cancel{\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial P}{\partial y}\right)}+2P\frac{\partial^2 Q}{\partial x\partial y}+2Q\frac{\partial^2 P}{\partial x\partial y}\right]dx\,dy$$

$$=\iint_D\left[2P\frac{\partial^2 Q}{\partial x\partial y}+2Q\frac{\partial^2 P}{\partial x\partial y}\right]dx\,dy$$

$$=2\iint_D\left[P\frac{\partial^2 Q}{\partial x\partial y}+Q\frac{\partial^2 P}{\partial x\partial y}\right]dx\,dy=\int_{\partial D}\left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)dx+\left(P\frac{\partial Q}{\partial y}-Q\frac{\partial P}{\partial y}\right)dy \checkmark$$

#15 Evaluate the line integral

$\int_C (2x^3-y^3)\,dx+(x^3+y^3)\,dy$ where C is the unit circle and verify Green's theorem for this case.

unit circle parametrization: $x=\cos\theta$, $y=\sin\theta$, $\theta\in[0,2\pi]$

$$\Rightarrow \int_0^{2\pi}\left((2\cos^3\theta-\sin^3\theta)(-\sin\theta)+(\cos^3\theta+\sin^3\theta)\cos\theta\right)d\theta$$

$$=\int_0^{2\pi}-2\cos^3\theta\sin\theta+\sin^4\theta+\cos^4\theta+\cos\theta\sin^3\theta\,d\theta$$

$\frac{3\pi}{2}$ $$=\int_0^{2\pi}\left[\sin^4\theta+\cos^4\theta\right]+\sin\theta\cos\theta\left[\sin^2\theta-2\cos^2\theta\right]d\theta$$

$$=\int_0^{2\pi}\left[(1-2\sin^2\theta\cos^2\theta)+\sin\theta\cos\theta\left[1-3\cos^2\theta\right]\right]d\theta$$

$$=\int_0^{2\pi}d\theta-\frac{1}{2}\int_0^{2\pi}\sin^2(2\theta)\,d\theta-\int_0^{2\pi}\sin\theta\cos\theta-\int_0^{2\pi}3\cos^3\theta\sin\theta\,d\theta$$

$$=2\pi-\frac{1}{2}\int_0^{2\pi}\frac{1-\cos(4\theta)}{2}d\theta+\left(\frac{1}{2}\cos^2\theta\right)\Big|_0^{2\pi}+\frac{(3\cos^4\theta)}{4}\Big|_0^{2\pi}$$

$$=2\pi+\left(-\frac{\theta}{4}+\frac{\sin 4\theta}{16}\right)\Big|_0^{2\pi}+0+0$$

$$=2\pi-\frac{2\pi}{4}=\boxed{\frac{3\pi}{2}}$$

#19 (a) $\vec{F} = x\hat{i} + y\hat{j}$ $\quad\gamma(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$

unit disk $x^2 + y^2 \le 1$

$$n = \frac{(\cos t, +\sin t)}{\sqrt{(-\sin t)^2 + (\cos t)^2}} = (\cos t, \sin t)$$

$$\int_{\partial D} (\cos t, \sin t)(\cos t, \sin t)\,ds$$

$$= \int_0^{2\pi} \cos^2 t + \sin^2 t \; ds = \int_0^{2\pi} 1 \; ds = 2\pi$$

Now $\iint_D \nabla \cdot \vec{F}\, dA = \iint_D \nabla \cdot (x, y)\, dA = \int_0^1 \int_0^{\pi} 1 + 1 \; dA = \int_0^1 2\pi = 2\pi$

$\therefore$ the divergence theorem holds for $\vec{F} = x\hat{i} + y\hat{j}$.

(b) Now $\vec{F} = 2xy\hat{i} - y^2\hat{j}$ around Ellipse defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\iint_D \nabla \cdot \vec{F}\, dA = \iint_D 2y - 2y \; dA = \boxed{0}$$

#20 $\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2)}\right) = \left(\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}\right) = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \left(\frac{-(x^2+y^2) + y(2y)}{(x^2+y^2)^2}\right) = \frac{y^2 - x^2}{(x^2+y^2)^2}$

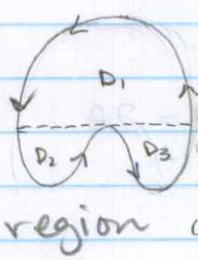
$$\iint_D \left[\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2}\right] dx\,dy = 0$$

#24 $x = r\cos\theta$ $\quad A = \frac{1}{2}\int_{\partial D} (r\cos\theta)(r\cos\theta) - (r\sin\theta)(-r\sin\theta)\,\theta$

$y = r\sin\theta$

$$\boxed{= \frac{1}{2}\int_a^b r^2 \, d\theta}$$

#25



I can use Green's Theorem to have that $D = D_1 \cup D_2 \cup D_3$

Then apply Green's Theorem to each region and sum the results.

#27 Use Green's Thrm to find the area of one loop of the 4 leafed rose $r = 3\sin 2\theta$. Hint: $x\,dy - y\,dx = r^2 d\theta$

$$\Rightarrow \text{then} \quad A = \frac{1}{2}\int r^2 d\theta$$

$$= \frac{1}{2}\int (9\sin^2(2\theta))\,d\theta = \frac{9}{2}\int \frac{1-\cos(4\theta)}{2}\,d\theta$$

$$= \frac{9}{4}\int 1-\cos(4\theta)\,d\theta = \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Big|_0^{\pi/2}$$

$$= \frac{9}{4}\left(\frac{\pi}{2} - 0\right) = \boxed{\frac{9\pi}{8}}$$

#28 Show that if C is a simple closed curve that bounds a region to which Green's thrm applies, then the area of the region D bounded by C is

$$A = \int_{\partial D} x\,dy = -\int_{\partial D} y\,dx$$

Area D be bounded by C is $A = \iint_D dx\,dy$

By Green's: $\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Let $P = 0$; $Q = x$

$$\Rightarrow \int_{\partial D} x\,dy = \iint_D dx\,dy = A$$

And $P = -y$, $Q = 0$

$$\Rightarrow \int_{\partial D} -y\,dx = \iint_D dx\,dy = A$$

$$\therefore \quad A = \int_{\partial D} x\,dy = -\int_{\partial D} y\,dx$$

Now $\int_{\partial D} x\,dy - y\,dx = \int_{\partial D} x\,dy - \int_{\partial D} y\,dx = A + A = 2A$

therefore, $A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx.$